RF Circuit Design

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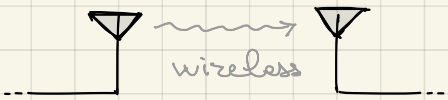
A.A. 2020/21

Communication Theory

How do we deliver an information?

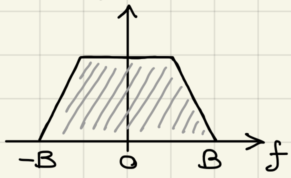


$\Longrightarrow$ Carrier modulation

sinusoid $\underbrace{A_c}_{\text{amplitude}} \cos\underbrace{(\omega_c t)}_{\text{phase}}$

"Carrier" because it carries the information.

Why do we need "modulation" instead of just transmitting the original information without carrier?



Baseband signal (i.e. original information) is typically centered around the origin.

Issue: $\frac{\lambda}{2}$ physical dimension of ideal Hertz dipole (antenna)

If $\frac{\lambda}{2} = 15$ cm → $\lambda = 30$ cm → $f_c = \frac{c}{\lambda} = 1$ GHz !

Modulation is needed because antennas work around a certain frequency that depends on their size.
Hence we need to move the signal information to such frequency using a carrier of that same frequency.

AM (Amplitude Modulation)

baseband signal

$$\boxed{x(t) = A_c\left[1 + m \cdot x_{BB}(t)\right] \cos(\omega_c t)}$$

Spectrum: Fourier transform of $x(t)$

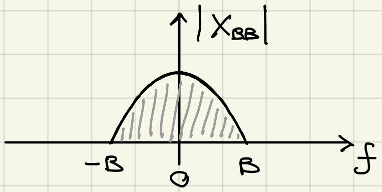
$$X(f) := \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$x \cdot y \xrightarrow{\mathcal{F}} X * Y$

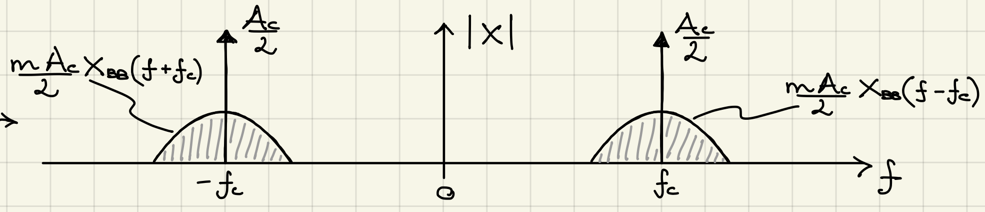
$e^{j2\pi f_c t} \xrightarrow{\mathcal{F}} \delta(f-f_c)$

$$x(t) = A_c\left[1 + m \cdot x_{BB}(t)\right] \frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}$$

$$\Longrightarrow X(f) = \frac{A_c}{2}\delta(f-f_c) + \frac{A_c}{2}\delta(f_c+f_c) + \frac{mA_c}{2} X_{BB}(f-f_c) + \frac{mA_c}{2} X_{BB}(f+f_c)$$



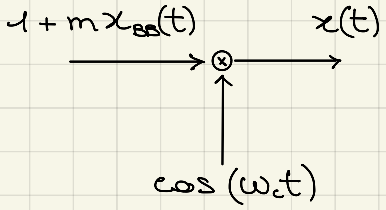
$\xrightarrow[AM]{f_c \gg B}$



After modulation (TX) we need demodulation (RX).

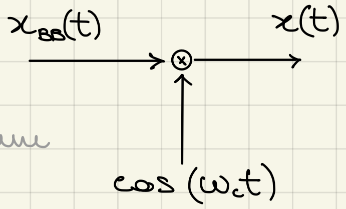
TX

- AM with transmitted carrier



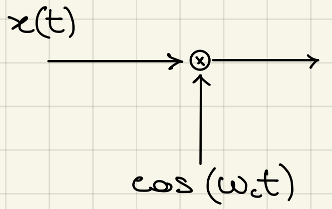
- AM without transmitted carrier

no deltas in the spectrum



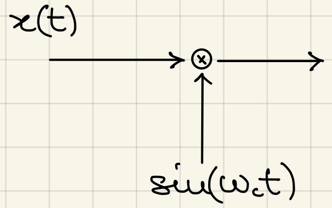
RX

- Coherent demodulation (without transmitted carrier)



$$x(t)\cos(\omega_c t) =$$
$$= x_{BB}(t)\cos^2(\omega_c t) = \longrightarrow \boxed{\text{LPF}} \longrightarrow \sim \frac{1}{2} x_{BB}(t)$$
$$= x_{BB}(t)\,\frac{1+\cos(2\omega_c t)}{2}$$

"Coherent" because the demodulating signal is in phase with the modulating signal.

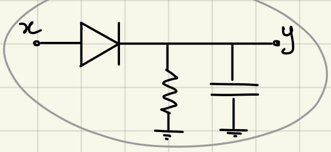


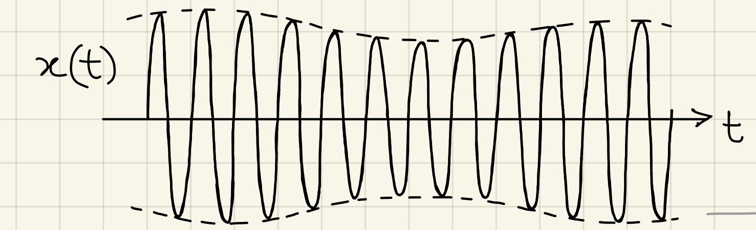
$$x(t)\sin(\omega_c t) =$$
$$= x_{BB}(t)\sin(\omega_c t)\cos(\omega_c t) = \longrightarrow \boxed{\text{LPF}} \longrightarrow \sim 0$$
$$= x_{BB}(t)\frac{1}{2}\sin(2\omega_c t)$$

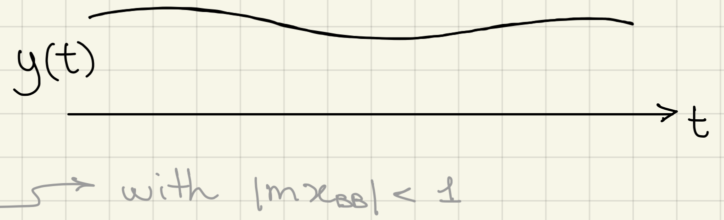
Issue: any phase error between transmitter and receiver will cause a degradation of the signal. This can be a problem since TX and RX have their own independent clocks that might have a synchronous mismatch.

- Non-coherent demodulation (with transmitted carrier)

$$x(t) = A_c[1 + m x_{BB}(t)]\cos\omega_c t \longrightarrow \boxed{\text{Envelop (peak) detector}} \longrightarrow y(t)$$



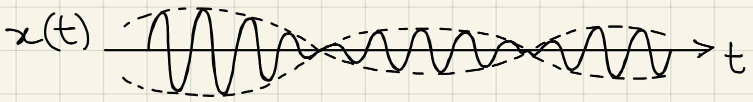


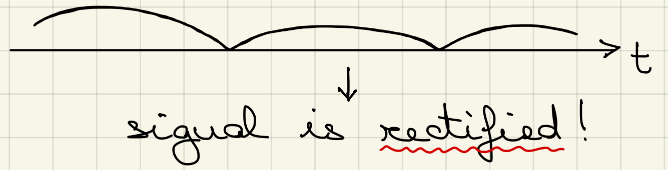


Advantage: RX does not need any internal clock for demodulation.

- Non-coherent demodulation (without transmitted carrier)

$$x(t) = x_{BB}(t)\cos\omega_c t \longrightarrow \boxed{\text{Env. det.}} \longrightarrow y(t)$$



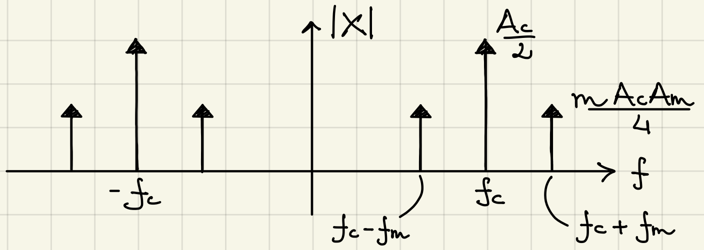


Issue: non-coherent dem. has to transmit the carrier, which impairs the efficiency of the process.

Phasor representation of a sinusoidal AM

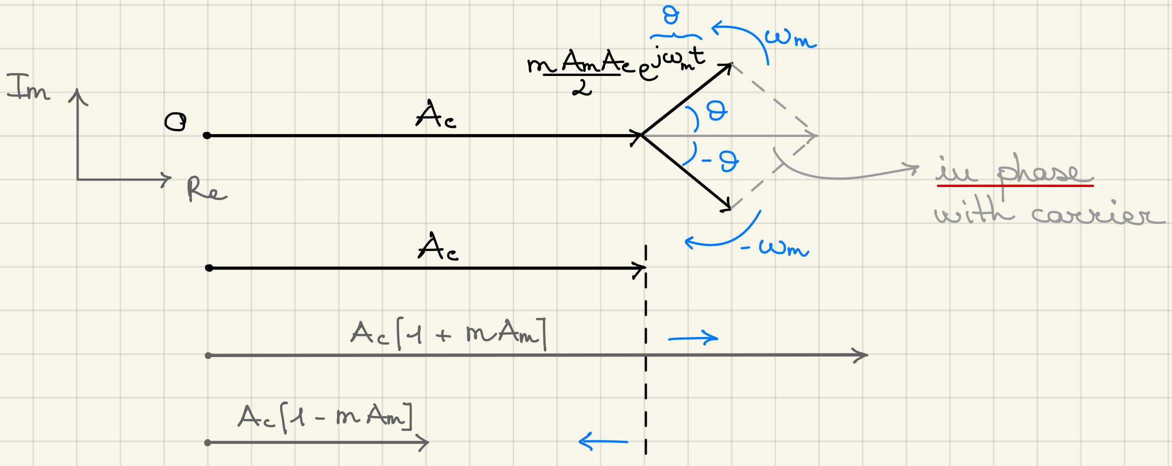
$$x_{BB}(t) = A_m \cos \omega_m t$$

$$x(t) = A_c[1 + m\, x_{BB}(t)] \cos\omega_c t =$$
$$= A_c \cos\omega_c t + m A_m A_c \cos\omega_c t \cos\omega_m t =$$
$$= A_c \cos\omega_c t + \frac{mA_mA_c}{2}\cos(\omega_c-\omega_m)t + \frac{mA_mA_c}{2}\cos(\omega_c+\omega_m)t$$



phasor

$$x(t) = \text{Re}\{\bar{X}(t)\, e^{j\omega_c t}\} \implies \bar{X}(t) = A_c + \frac{mA_mA_c}{2}\left[e^{-j\omega_m t} + e^{j\omega_m t}\right]$$



PM
‖

FM (Frequency Modulation)

$$\boxed{x(t) = A_c \cos\Big[\omega_c t + \overbrace{m\int_{-\infty}^{t} x_{BB}(t')\,dt'}^{\varphi(t)}\Big]}$$

$$\begin{cases} \omega(t) = \dfrac{d\varphi}{dt} \\ \varphi(t) = \displaystyle\int_{-\infty}^{t} \omega(t')\,dt' \end{cases}$$

Relationship between angular frequency ω and phase φ of a periodic signal

Narrow Band FM approximation (NBFM):

$$\boxed{\varphi(t) = m\int_{-\infty}^{t} x_{BB}(t')\,dt' \ll 1 \text{ rad}}$$

$$x(t) = A_c\cos[\omega_c t + \varphi(t)]$$

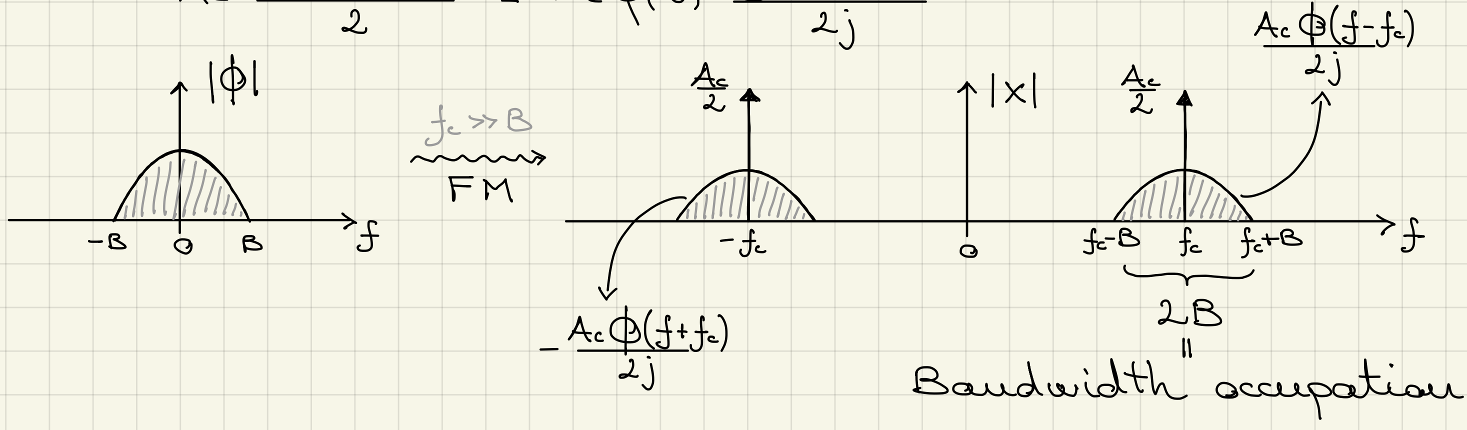
$$= A_c\cos\omega_c t \cdot \cos[\varphi(t)] - A_c \sin\omega_c t \cdot \sin[\varphi(t)]$$

NBFM

$$\approx A_c\cos\omega_c t \cdot 1 - A_c\sin\omega_c t \cdot \varphi(t)$$

$$= \underbrace{A_c\cos\omega_c t}_{\text{carrier}} - \underbrace{A_c\varphi(t)\sin\omega_c t}_{\text{AM modulation of the quadrature component of the carrier}}$$

$$= A_c\,\frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2} - A_c\,\varphi(t)\cdot\frac{e^{j\omega_c t} - e^{-j\omega_c t}}{2j}$$



Case: sinusoidal FM.

It is possible in this case to study the spectrum with no approximation.

$$x_{BB}(t) = A_m\cos\omega_m t$$

$$x(t) = A_c\cos\Big[\omega_c t + m\int_{-\infty}^{t} A_m\cos\omega_m t'\,dt'\Big] =$$

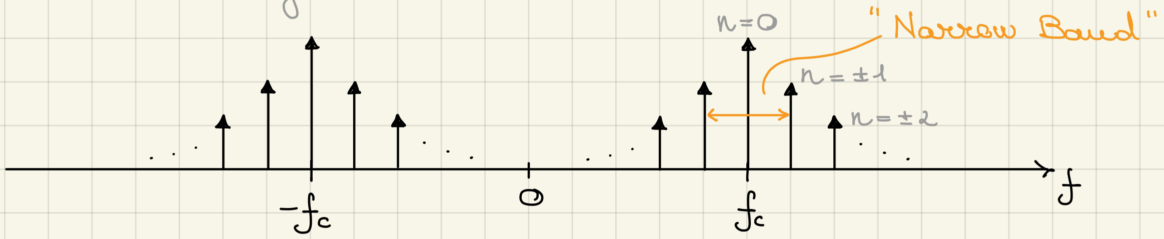
$$= A_c\cos\Big[\omega_c t + \overbrace{\boxed{\frac{mA_m}{\omega_m}}\sin\omega_m t}^{\varphi(t)}\Big] \rightarrow \text{"modulation depth" } \beta$$

$$x(t) = A_c \cos[\omega_c t - \beta \sin \omega_m t]$$

write cos(sin t) as a Fourier series

$$= A_c \sum_{n=-\infty}^{+\infty} \left\{ \boxed{J_n(\beta)} \cdot \cos[\omega_c + n\omega_m t] \right\}$$

first kind Bessel function



The bandwidth occupation of the entire signal would be infinite, due to the non-linearity of the modulation without approximation.

Carson's BandWidth:

NBFM ⟷ $\varphi \ll 1 \text{rad}$
$\beta \ll 1 \text{rad}$

Bandwith associated to 98% of the energy of the FM carrier

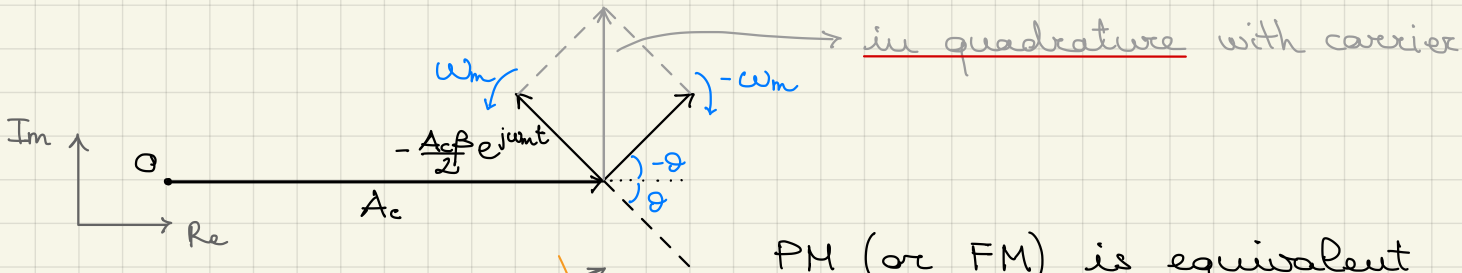
$$\boxed{BW_{98\%} = 2(\beta+1) f_m} \approx 2 f_m$$

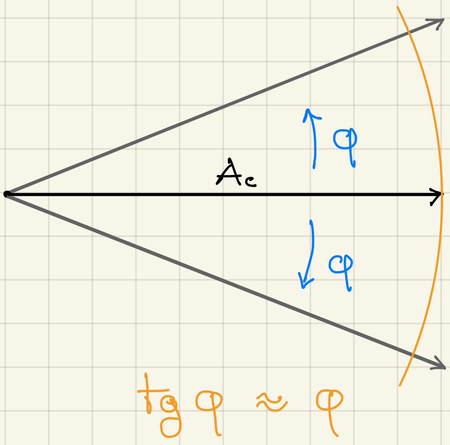
bandwidth of the baseband signal

Phasor representation of a sinusoidal FM

$$x(t) = A_c \cos[\omega_c t + \varphi(t)] \approx A_c \cos \omega_c t - A_c \varphi(t) \cdot \sin \omega_c t$$
$$= A_c \cos \omega_c t - A_c \sin \omega_c t \cdot [-\beta \sin \omega_m t] =$$
$$= A_c \cos \omega_c t + \frac{A_c \beta}{2} \cos(\omega_c - \omega_m)t - \frac{A_c \beta}{2} \cos(\omega_c + \omega_m)t$$

$$\Rightarrow \overline{X}(t) = A_c + \frac{A_c \beta}{2} \cdot \left[e^{-j\omega_m t} - \overset{\theta}{\overline{e^{j\omega_m t}}} \right]$$





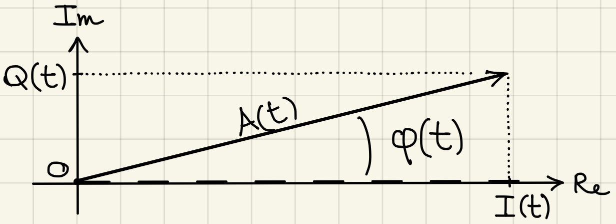
PM (or FM) is equivalent to amplitude modulation of the carrier.

Then why is it not a pure PM modulation?

The equivalence holds only under NBFM approximation $\beta \ll 1 \text{rad}$ (or $\varphi \ll 1 \text{rad}$)

AM and PM (Quadrature Modulation)

- $x(t) = a(t)\cos[\omega_c t + \varphi(t)]$



Phasor: $\bar{X}(t) = A(t)\,e^{j\varphi(t)}$

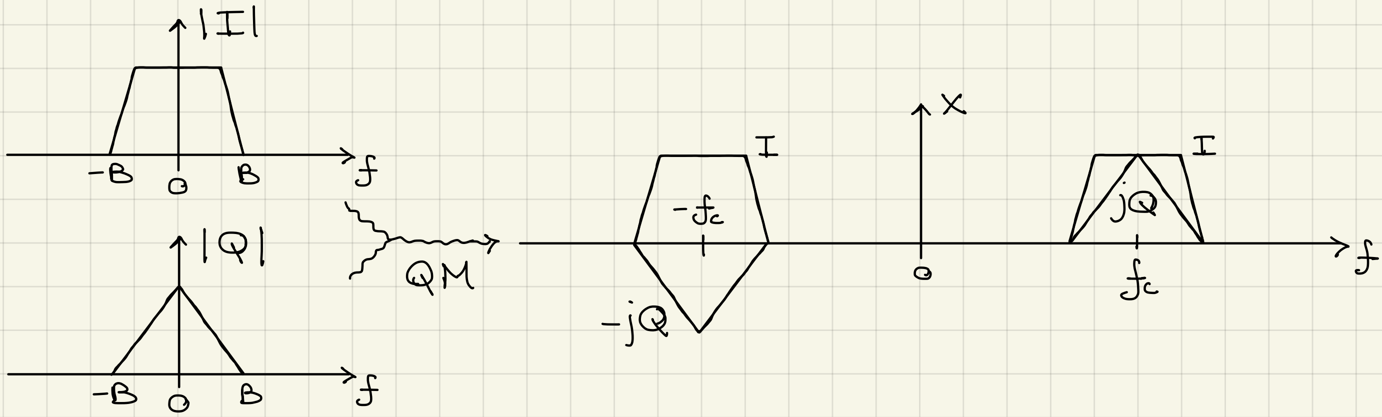
$$\mathrm{Re}\{\bar{X}(t)e^{j\omega_c t}\} = \mathrm{Re}\{A(t)\cos[\omega_c t+\varphi(t)] + jA(t)\sin[\omega_c t+\varphi(t)]\}$$

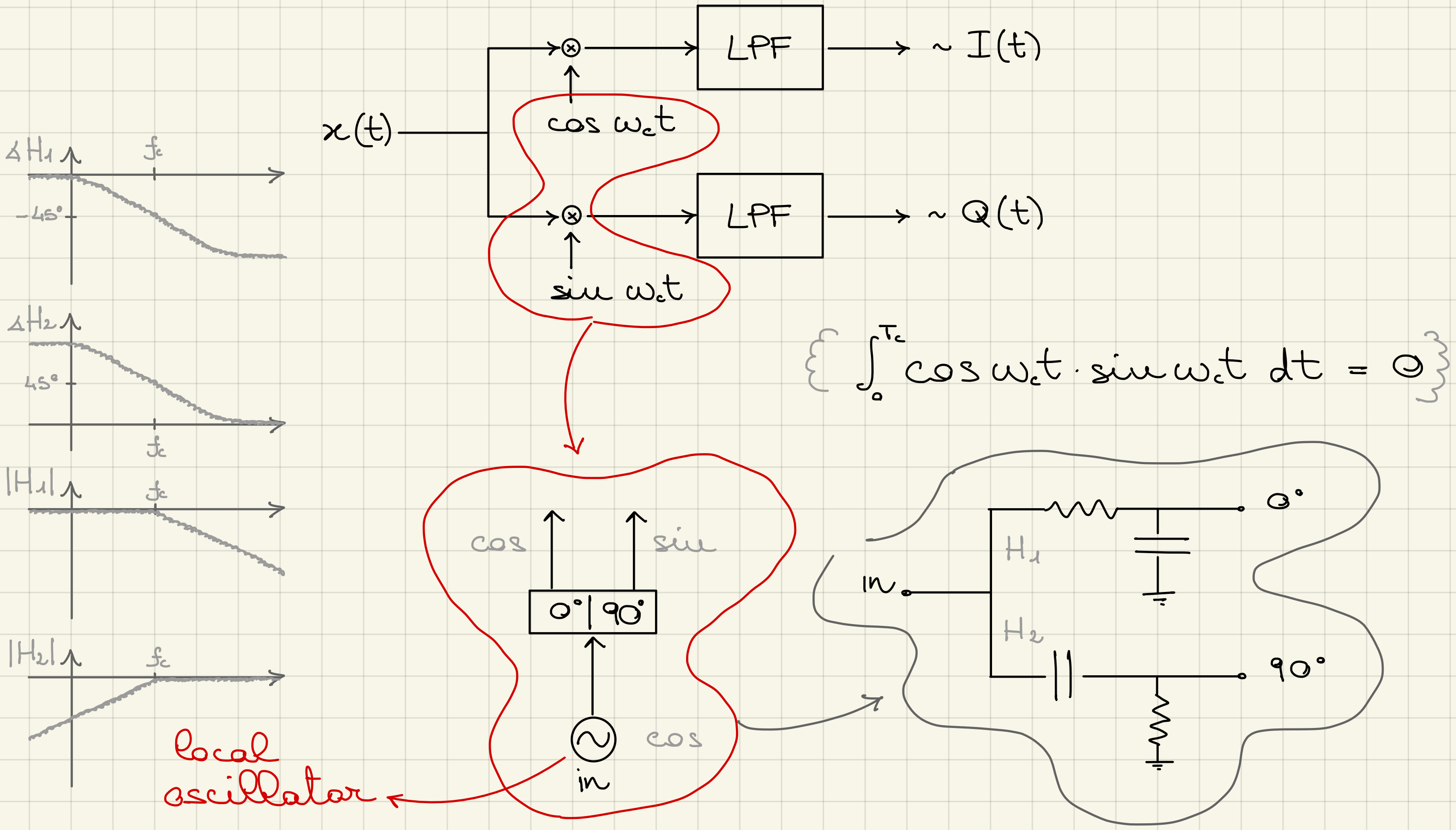
- $x(t) = I(t)\cos\omega_c t - Q(t)\sin\omega_c t =$

$$= I(t)\,\frac{e^{j\omega_c t}+e^{-j\omega_c t}}{2} + jQ(t)\,\frac{e^{j\omega_c t}-e^{-j\omega_c t}}{2} =$$

$$= \frac{1}{2}\underbrace{[I(t)+jQ(t)]}_{}\,e^{j\omega_c t} + \frac{1}{2}[I(t)-jQ(t)]\,e^{-j\omega_c t}$$

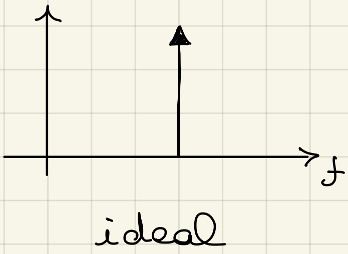
$$= \frac{1}{2}\,\bar{X}(t)\,e^{j\omega_c t} + \frac{1}{2}\,\bar{X}^*(t)\,e^{-j\omega_c t} = 2\cdot\cancel{\frac{1}{2}}\,\mathrm{Re}\{\bar{X}(t)\,e^{j\omega_c t}\}$$





$$\left\{\int_0^{T_c}\cos\omega_c t\cdot\sin\omega_c t\,dt = 0\right\}$$

We are not able to build a pure ideal oscillator:



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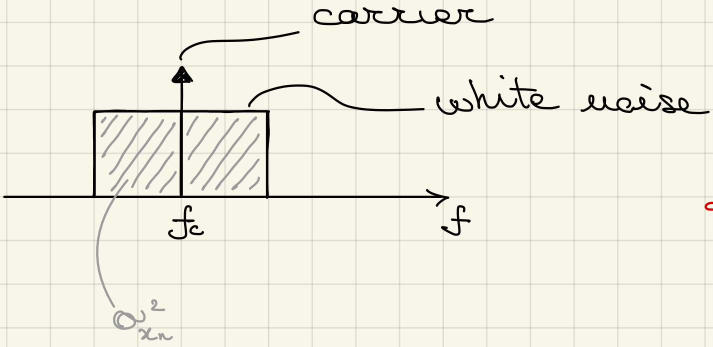
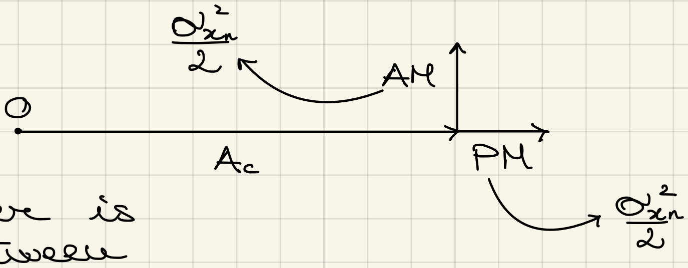
"spurs", unwanted tones, unwanted linewidth, phase noise...



$$A_c \cos\omega_c t + x_n(t) =$$
$$= A_c(1 + a_n(t))\cos[\omega_c t + \varphi_n(t)]$$



Rice theorem:

White noise power is equally split between phase noise (PM) and amplitude noise (AM)

We generally do not worry about amplitude noise whilst we do care about phase noise, for the following reasons:

- there is usually clamping of the signal/carrier which removes any amplitude fluctuation

  phase noise can come from the integration of frequency noise:

$$\varphi_n(t) = \int_{-\infty}^{t} \omega_n(t')dt'$$

which in terms of PSD means:

$$S_{\varphi_n} = \frac{1}{4\pi^2 f^2} S_{\omega_n}$$

if S_{ω_n} = const. then Random Walk

which diverges at low frequencies for white noise, that is at long observation times.
Amplitude noise does not suffer from this issue.

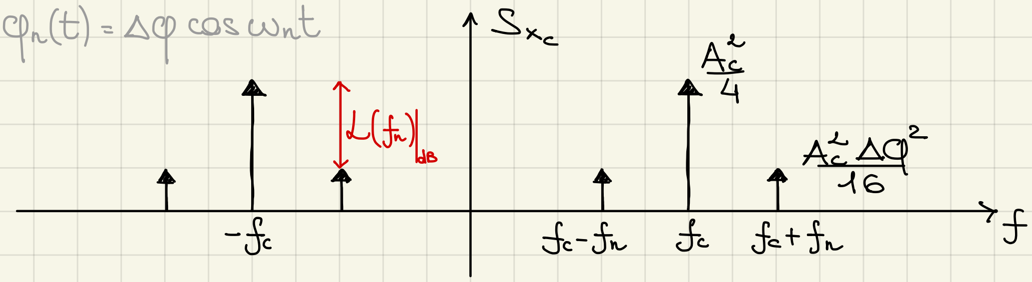
Consider phase noise as a sinusoidal disturb:

neglect $a_n(t)$ $\qquad$ $|\varphi_n(t)| \ll 1 \leftrightarrow$ NBFM

$$x_c(t) \approx A_c\cos[\omega_c t + \varphi_n(t)] \approx A_c\cos\omega_c t \cdot 1 - A_c\sin\omega_c t\,\varphi_n(t) =$$

$$= A_c\cos\omega_c t - \frac{A_c\Delta\varphi}{2}\cos(\omega_c+\omega_n)t + \frac{A_c\Delta\varphi}{2}\cos(\omega_c-\omega_n)t$$

$\varphi_n(t) = \Delta\varphi\cos\omega_n t$



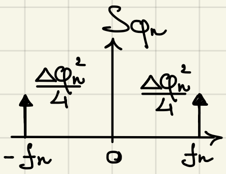
Single Sideband to Carrier Ratio (SSCR):

$$\boxed{\mathcal{L}(f_n) := \frac{S(f_c+f_n)}{P(f_c)}} \approx \frac{A_c^2\frac{\Delta\varphi^2}{16}}{\frac{A_c^2}{4}} = \frac{\Delta\varphi^2}{4}$$

frequency offset

NBFM, sinusoidal noise

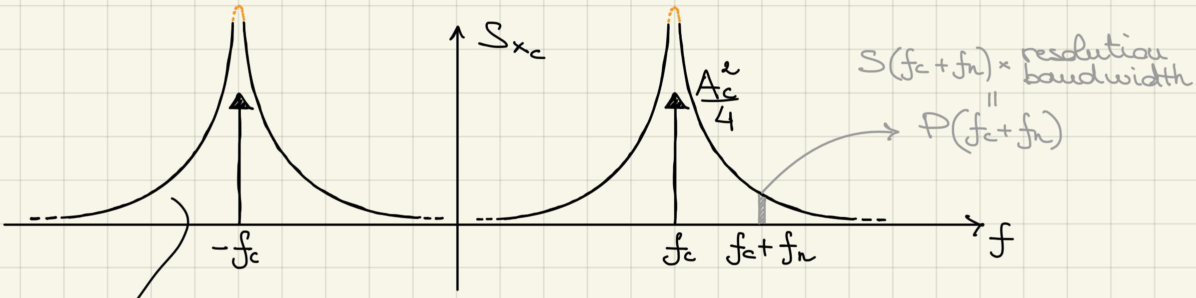
=



$$\boxed{\mathcal{L}(f_n) = \frac{S^{SSB}_{\varphi_n}(f_n)}{2}} \Longleftarrow S^{DSB}_{\varphi_n}(f_n)$$

This result can be extended to any noise shape.

Example: phase noise as random walk



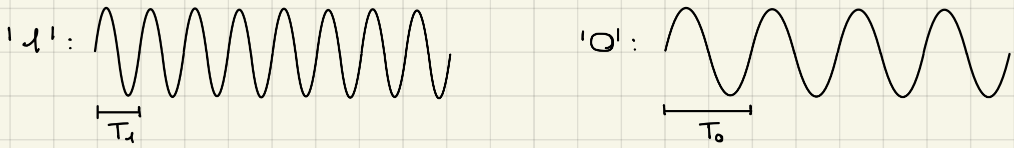
goes as $1/f^2$ so it should diverge at $f = f_c$!

However if $f = f_c$ then $\varphi_n \gg 1$ and NBFM does not hold anymore!

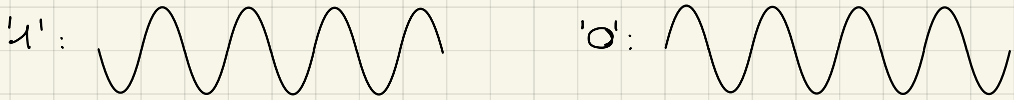
With no approximation it can be demonstrated that the noise has a Lorentzian shape around f_c.

Digital Modulation

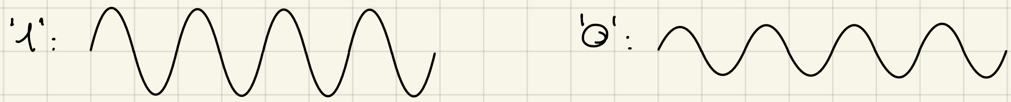
- FSK (Frequency Shift Keying)



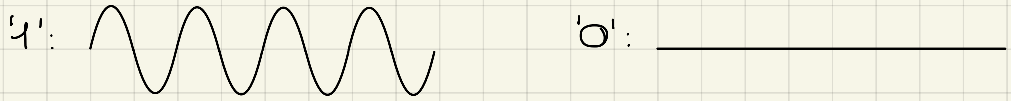
BPSK (Binary Phase Shift Keying)



- ASK (Amplitude Shift Keying)



- OOK (On Off Keying)



Digital modulation is <u>not</u> only binary. Using more symbols allows to have a higher bit rate at the same symbol (transmission) rate.

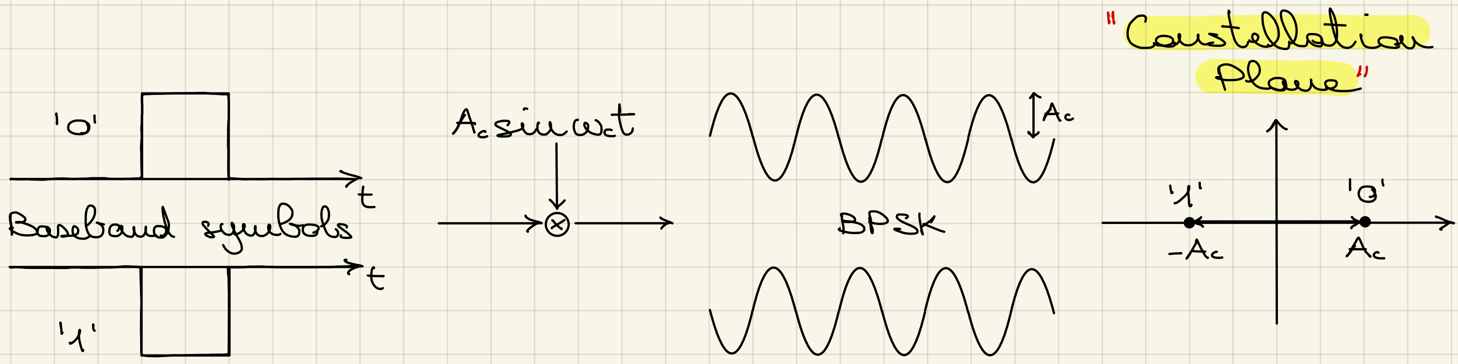
Shannon's capacity theorem (AWGN channel)

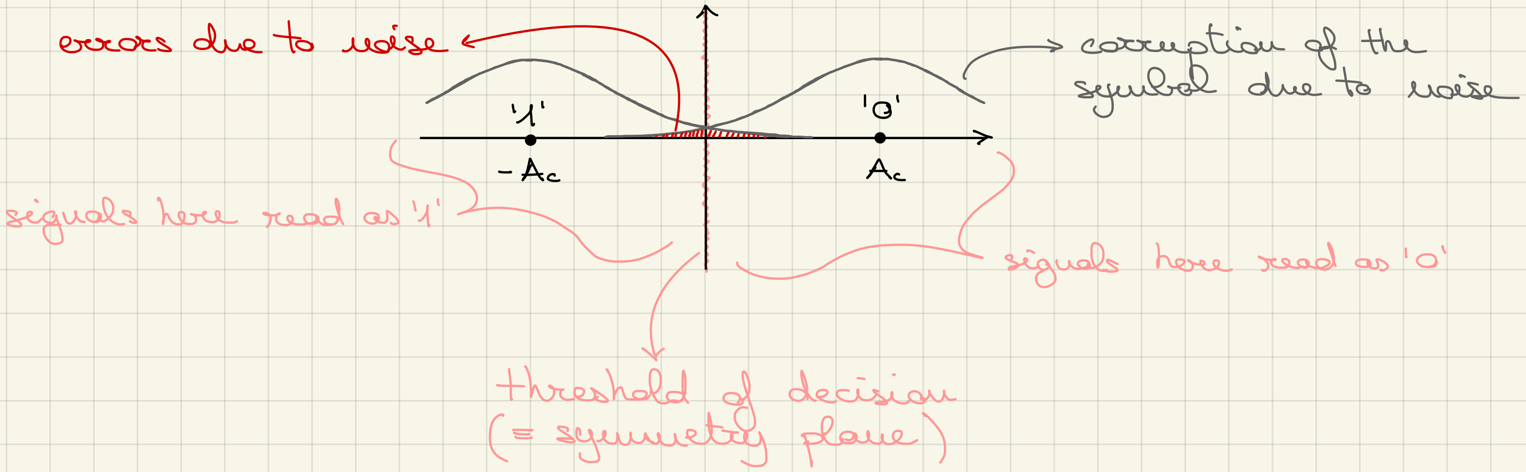
Additive White Gaussian Noise

$$\left[\, C = B \log_2\left(1 + \frac{P_s}{P_n}\right) \right]$$

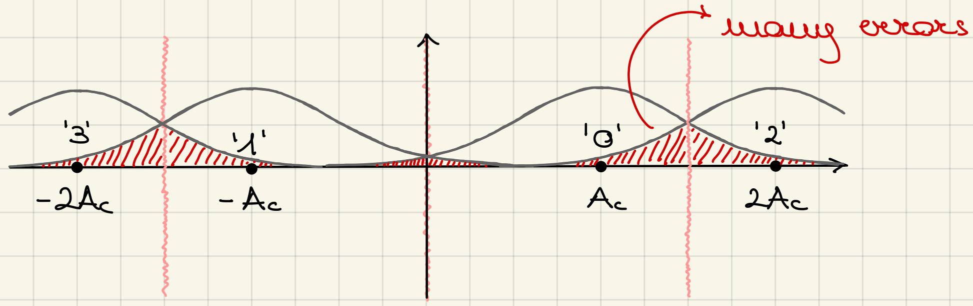
SNR

(maximum) bit-rate [bit/s]   Bandwidth [Hz]





To avoid errors, either signal power (A_c) should increase or noise power should decrease. So a faster bit-rate (fewer transmission errors) is granted by a higher SNR, which is what Shannon's theorem on channel capacity says.



Note how using more symbols, which should in theory increase the bit-rate, does not really improve it unless a higher signal power is adopted, since otherwise the bit-rate is impaired by transmission errors.
In fact, the number of symbols does not appear in Shannon's capacity theorem, hence just using more symbols won't improve the bit-rate.

Digital modulation: $\boxed{x_{BB}(t) = \sum_{n=-\infty}^{+\infty} b_n \, p(t - nT_b)}$

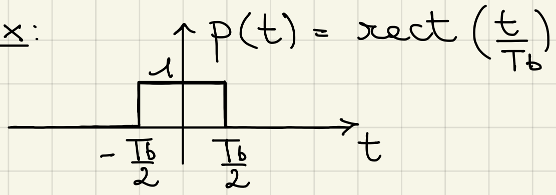
$\frac{1}{T_b}$: bit-rate

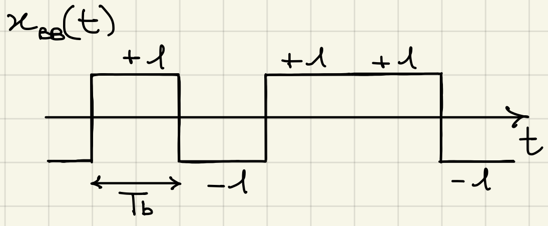
$p(t)$: pulse (symbol) shape

$b_n = \pm 1$ binary modulation

$b_n = \pm 1, \pm 2, \dots, \pm M$ multi level or M-ary modulation

Ex:



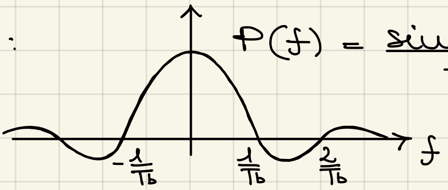


Now what is the bandwidth (B) of $x_{BB}(t)$?

{ b_n is random, x_{BB} is a stochastic process }

Theorem: $\boxed{S_{x_{BB}}(f) = \frac{|P(f)|^2}{T_b} \text{ where } P(f) = \mathcal{F}\{p(t)\}}$

Ex:



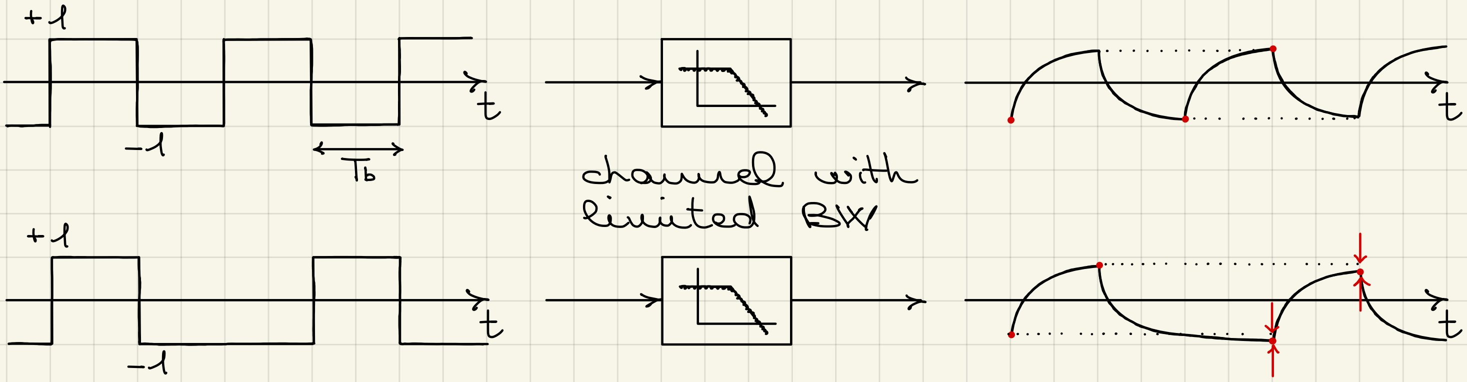
$$P(f) = \frac{\sin(\pi f T_b)}{\pi f} = T_b \operatorname{sinc}(f T_b) = \mathcal{F}[\underbrace{\operatorname{rect}(\tfrac{t}{T_b})}_{\substack{\| \\ p(t)}}]$$

$$\Longrightarrow S_{x_{BB}}(f) = \frac{|P(f)|^2}{T_b} = \frac{T_b^2 \operatorname{sinc}^2(f T_b)}{T_b} = T_b \operatorname{sinc}^2(f T_b)$$

"BW of $x_{BB}(t)$ is the same as BW of $p(t)$"

$$\text{Power of } x_{BB}(t) = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} \underbrace{x_{BB}^2(t)}_{= +1\ \forall t} dt = 1$$

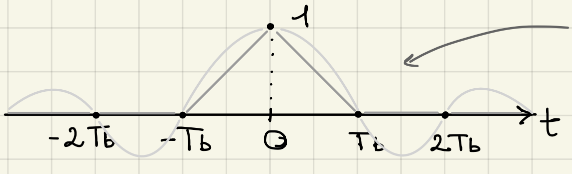
Issue: Intersymbol Interference (ISI)



If a symbol lasts longer than T_b, then it will pile up with the following symbols.
It degrades the SNR.

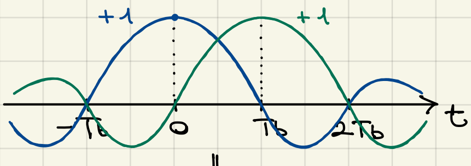
Solution: Nyquist signaling

$x_{BB}(t) = \sum_{n=-\infty}^{\infty} b_n\, p(t - nT_b)$, $p(t)$ such that $p(kT_b) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$



any shape respecting this conditions will work

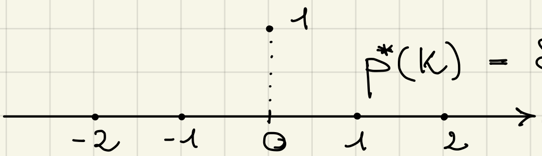
$$x_{BB}(t) = \underset{+1}{\underset{\shortparallel}{b_0}}\, p(t) + \underset{+1}{\underset{\shortparallel}{b_1}}\, p(t-T_b) + \dots$$



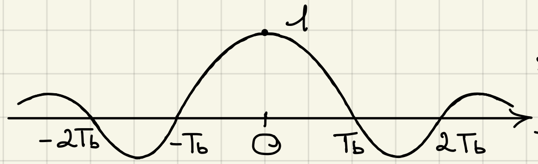
$\Downarrow$

NO ISI

Spectrum of Nyquist signal



$$p^*(k) = \delta(k) \xrightarrow{\mathcal{F}} P^*(f) = 1$$

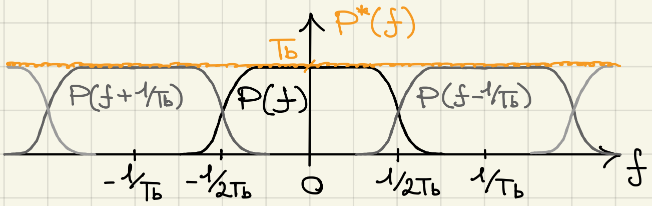


$$p(t) \xrightarrow{\mathcal{F}} P(f)$$

$$\Longrightarrow p^*(k) = p(t) \sum_k \delta(t - kT_b) \xrightarrow{\mathcal{F}} P(f) * \frac{1}{T_b} \sum_k \delta\left(f - \frac{k}{T_b}\right) =$$

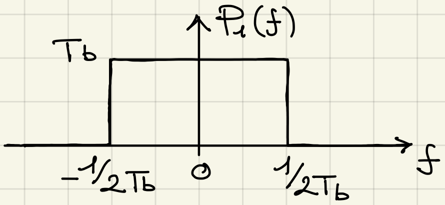
$$= \frac{1}{T_b} \sum_k P\left(f - \frac{k}{T_b}\right) = P^*(f)$$

$$\Longrightarrow \left[\sum_{k=-\infty}^{+\infty} P\left(f - \frac{k}{T_b}\right) = T_b \right]$$

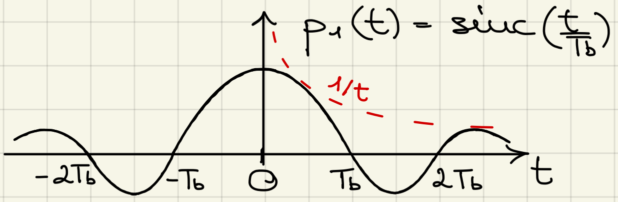


Examples:

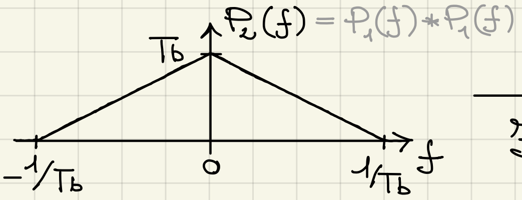
1)



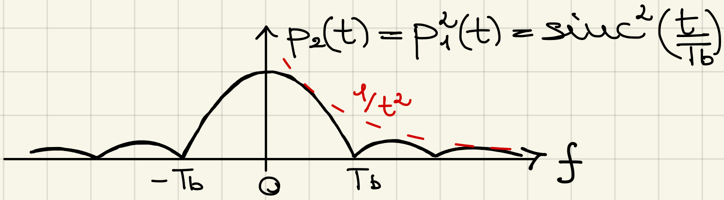
$$\xrightarrow{\mathcal{F}^{-1}}$$



2)

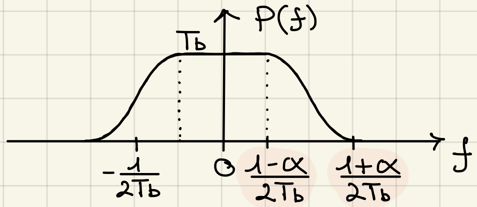


$$\xrightarrow{\mathcal{F}^{-1}}$$



3) "Raised cosine"

$$P(f) = 1 + \cos\left(\frac{\pi T_b}{\alpha}\right)\left(|f| - \frac{1-\alpha}{2T_b}\right)\frac{T_b}{2}$$



$$\xrightarrow{\mathcal{F}^{-1}}$$

$$p(t) = \text{sinc}\left(\frac{t}{T_b}\right) \frac{\cos\left(\pi\alpha\frac{t}{T_b}\right)}{1 - 4\alpha^2\left(\frac{t}{T_b}\right)^2}$$

α : roll-off factor $(0 \leqslant \alpha \leqslant 1)$

$\alpha = 0$: narrow spectrum (rect shape) $BW = \frac{1}{2T_b}$
slow envelop $(\div \frac{1}{t})$

$\alpha = 1$: wide spectrum (triang shape) $BW = \frac{1}{T_b}$
fast envelop $(\div \frac{1}{t^2})$

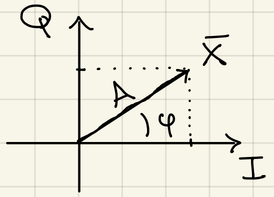
Even though $\alpha = 0$ is in theory preferable (for its narrower spectrum), the faster envelop of $\alpha = 1$ allows to reduce ISI when there are synchronization errors between symbols, since the signal interference (its value outside the peak) will be lower

$\Longrightarrow$ Trade-off between bandwidth occupation and resilience to synchronization errors.

Non-idealities of a Local Oscillator (LO)

Modulated signal: $x(t) = I(t)\cos\omega_c t - Q(t)\sin\omega_c t$ cartesian

$$= A(t)\cos[\omega_c t + \varphi(t)] \quad \text{polar}$$



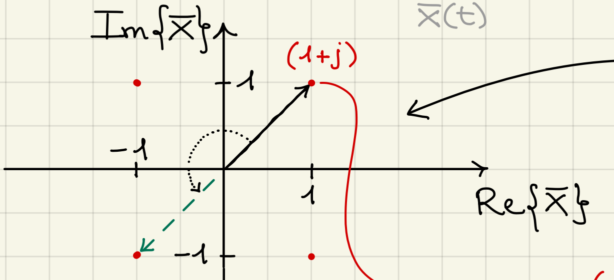
$$A(t) = \sqrt{I(t)^2 + Q(t)^2} \qquad \varphi(t) = \text{arctg}\,\frac{Q(t)}{I(t)}$$

We have already seen the effects of phase noise on the SSCR.

Let's now consider digital QPSK (Quadrature PSK):

$$x(t) = \overbrace{\sum_n a_n p(t-nT_b)}^{I(t)} \cos\omega_c t - \overbrace{\sum_n b_n p(t-nT_b)}^{Q(t)} \sin\omega_c t$$

$$= \text{Re}\Big\{\underbrace{\sum_n (a_n + j b_n)\cdot p(t-nT_b)}_{\bar{x}(t)} e^{j\omega_c t}\Big\} \qquad (a_n = \pm 1,\ b_n = \pm 1)$$



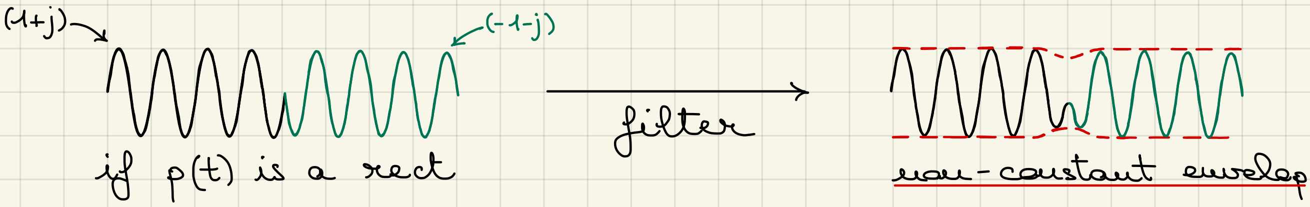
$t = kT_b \qquad p(0) = 1$

$\bar{X}(kT_b) = a_k + j b_k$

$x(t) = \cos\omega_c t + \sin\omega_c t$
if p(t) is a rect in time

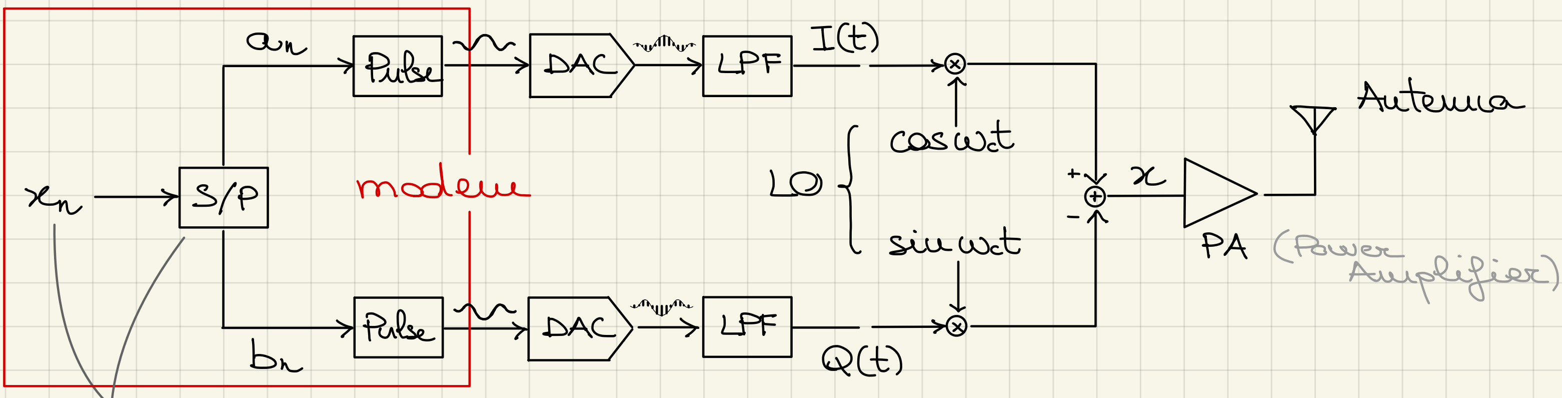
Constellation plane

Apparently, QPSK modulated signal seems to have a constant envelop (phasor has constant absolute value). However, non-instantaneous transitions between two symbols actually cause the envelop to be non-constant:

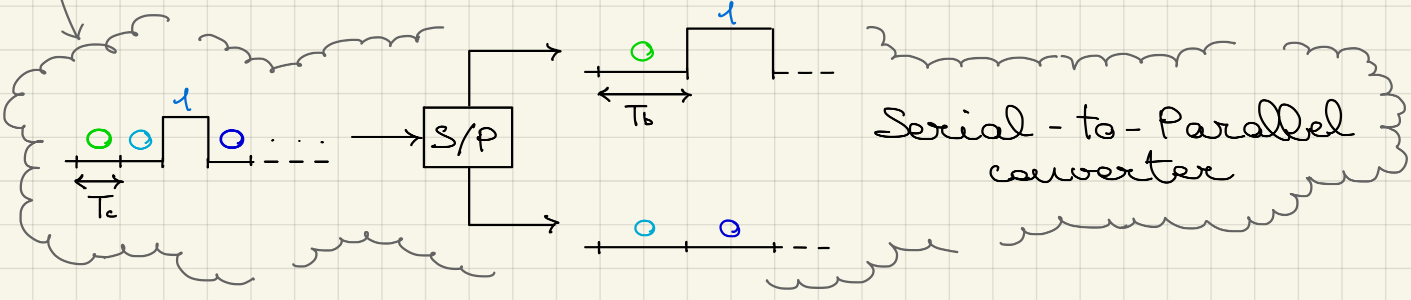


The envelop will be useful later # to describe the effects of some non-idealities.

QPSK TX block diagram



T_c chip period, $1/T_c$ chip rate, $T_c = T_b/2$



RF bandwidth:

For $\alpha = 0$ (roll-off) t-shape is $\mathrm{sinc}\left(\frac{t}{T_b}\right)$

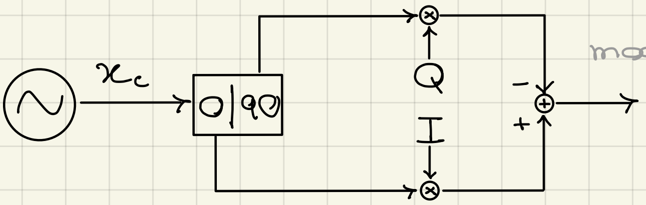
$$BW_{BB} = \frac{1}{2T_b} \Rightarrow BW \underset{\uparrow\,2BW_{BB}}{=} \frac{1}{T_b} = \frac{1}{2T_c}$$

RF bandwidth of QPSK is given by the chip frequency divided by 2

For $\alpha = 1$ then RF bandwidth is exactly the chip rate.

We can now see the impact of LO PHASE NOISE (and other issues) on the quality of the modulation.

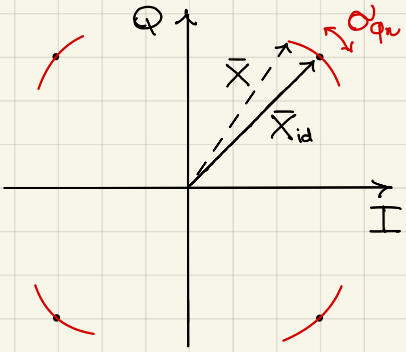
$$x_c(t) = \cos[\omega_c t + \varphi_n(t)]$$



modulated carrier

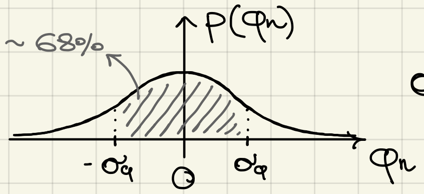
$$x(t) = Re\{\bar{X}_{id}(t)\, e^{j\omega_c t}\, e^{j\varphi_n(t)}\}$$

$$\bar{X}_{id}(t) = I(t) + jQ(t)$$



Phasor affected by LO phase noise:

$$\bar{X}(t) = \bar{X}_{id}(t)\, e^{j\varphi_n(t)}$$

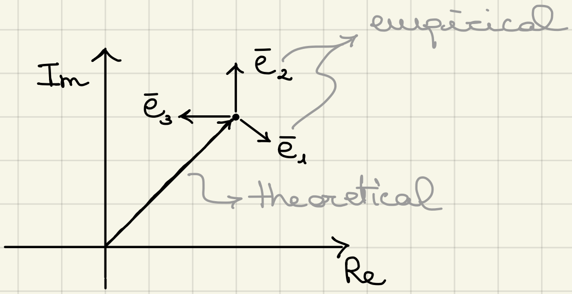


$$\sigma_{\varphi_n}^2 = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \varphi_n^2(t)\, dt = \int_0^{+\infty} S_{\varphi_n}^{SSB}(f)\, df$$

We introduce the following parameter:

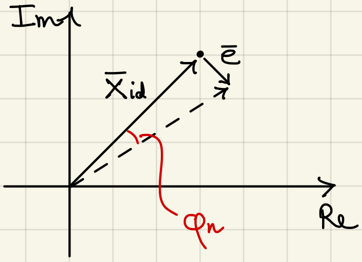
Error-Vector Magnitude

$$\boxed{EVM := \frac{\frac{1}{N}\sum_{k=1}^{N} |\bar{e}_k|^2}{P_{avg}}}$$



It is a noise-to-signal ratio: $EVM \sim \frac{1}{SNR}$

- EVM induced by phase noise:



$$\boxed{EVM = \frac{|\bar{e}|^2}{P_{avg}} \simeq \frac{|\bar{X}_{id}|^2 \cdot \sigma_\varphi^2}{|\bar{X}_{id}|^2} = \sigma_\varphi^2}$$

$$\uparrow$$

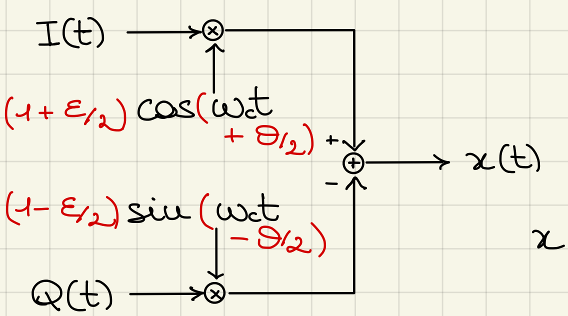
$$\underbrace{|\bar{e}|}_{chord} \simeq \underbrace{|\bar{X}_{id}|\,\varphi_n}_{arc} \Rightarrow |\bar{e}|^2 = |\bar{X}_{id}|^2\,\sigma_\varphi^2$$

$\Longrightarrow$ $EVM \simeq \sigma_\varphi^2$ regardless of P_{avg} (TX power).
SNR at TX output is limited by phase noise.

Also RX phase noise (LO) ⟶ degrades SNR at RX

$$SNR \leq \frac{1}{\sigma_\varphi^2} \quad (\text{"bottleneck"})$$

- EVM induced by amplitude/phase errors: ≠ noise



ε amplitude error

θ phase error

$$x(t) = \left(1+\frac{\varepsilon}{2}\right)\cos\left(\omega_c t + \frac{\theta}{2}\right)\cdot I(t) -$$
$$- \left(1-\frac{\varepsilon}{2}\right)\sin\left(\omega_c t - \frac{\theta}{2}\right)\cdot Q(t)$$

$$EVM = \frac{P_e}{P_{avg}} = \frac{|\bar{e}|^2}{|\bar{X}_{id}|} \qquad \bar{e} = \bar{X} - \bar{X}_{id}$$

$$\bar{X}_{id} = I + jQ \qquad \bar{X} = I e^{j\theta/2}\left(1+\frac{\varepsilon}{2}\right) + jQ e^{-j\theta/2}\left(1-\frac{\varepsilon}{2}\right)$$

$$\Longrightarrow -\bar{e} = \bar{X}_{id} - \bar{X} = I\Big[\underbrace{1 - e^{j\theta/2}}_{\sim -j\frac{\theta}{2}} - \underbrace{e^{j\theta/2}}_{\sim(1+j\theta/2)}\cdot\frac{\varepsilon}{2}\Big] + jQ\Big[\underbrace{1 - e^{-j\theta/2}}_{\sim +j\theta/2} + \underbrace{e^{-j\theta/2}}_{\sim(1-j\frac{\theta}{2})}\frac{\varepsilon}{2}\Big]$$

small θ

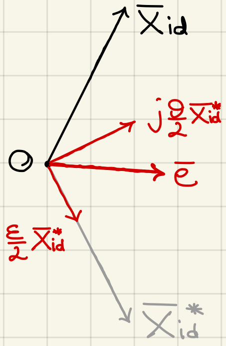
$e^x \simeq 1+x$
$1-e^x \simeq -x$
for $x \simeq 0$

$$\simeq I\left[-j\frac{\theta}{2} - \left(1+\cancel{j\frac{\theta}{2}}\right)\frac{\varepsilon}{2}\right] + jQ\left[j\frac{\theta}{2} + \left(1-\cancel{j\frac{\theta}{2}}\right)\frac{\varepsilon}{2}\right]$$

small ε → $\varepsilon\theta \ll \theta$

$$\simeq I\left[-j\frac{\theta}{2} - \frac{\varepsilon}{2}\right] + jQ\left[j\frac{\theta}{2} + \frac{\varepsilon}{2}\right] =$$

$$= -\left[j\frac{\theta}{2} + \frac{\varepsilon}{2}\right](I - jQ)$$



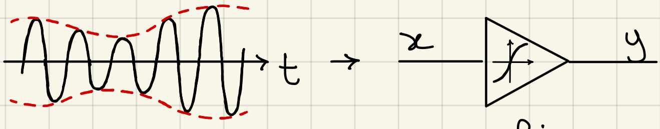
$$\Longrightarrow \bar{e} = \left[\frac{\varepsilon}{2} + j\frac{\theta}{2}\right]\bar{X}_{id}^*$$

$$\left[EVM = \frac{|\bar{e}|^2}{|\bar{X}_{id}|^2} = \frac{\left|\left(\frac{\varepsilon}{2} + j\frac{\theta}{2}\right)\bar{X}_{id}^*\right|^2}{|\bar{X}_{id}|^2} = \left(\frac{\varepsilon^2}{4} + \frac{\theta^2}{4}\right)\frac{\cancel{|\bar{X}_{id}^*|^2}}{\cancel{|\bar{X}_{id}|^2}}\right]$$

e.g.: $\varepsilon = 1\%$ $\theta = 1\ \text{deg} = 0{,}01704\ \text{rad}$

$$EVM = \left(\frac{0{,}01}{2}\right)^2 + \left(\frac{0{,}01704}{2}\right)^2 = 0{,}0004 \qquad EVM_{dB} = -33{,}9\ dB$$

- Impact of non-linearity on the modulated signal:



→ spectral regrowth

non-constant envelope

non-linear amplification

new frequency components outside bandwidth of interest

Remember:

$$\cos^3 x = \cos x \cdot \cos^2 x = \cos x \, \frac{1+\cos 2x}{2} =$$

$$= \frac{1}{2}\cos x + \frac{1}{2}\left[\frac{1}{2}\cos x + \frac{1}{2}\cos 3x\right] =$$

$$= \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$$

$$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

$$y(t) = \alpha_1 x(t) + \underbrace{\alpha_3 x^3(t)}_{\text{cubic non-linearity}} + \dots$$ "static non-linear model"

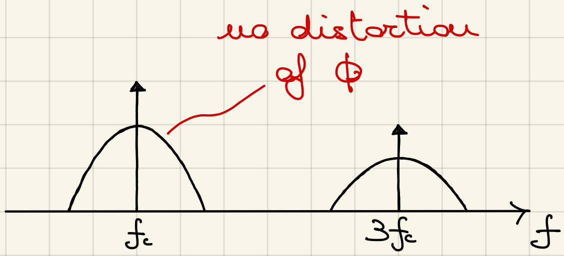
assuming no even order distortions (which are anyway less harmful than odd order in terms of spectral regrowth)

- Constant envelop: $x(t)$ is PM modulation

$$x(t) = \underset{\text{constant}}{\underset{\downarrow}{A_c}} \cos\left[\omega_c t + \underbrace{\varphi(t)}_{\text{information signal}}\right]$$

$$\alpha_3 x^3(t) = \alpha_3 A_c^3 \cos^3[\omega_c t + \varphi(t)] = \alpha_3 A_c^3 \frac{3}{4}\cos[\underline{\underline{\omega_c t}} + \underline{\underline{\varphi(t)}}] +$$

$$+ \alpha_3 A_c^3 \frac{1}{4}\cos[3\omega_c t + 3\varphi(t)]$$



A constant envelop signal is not affected by non-linear amplifications.

- Non-constant envelop:

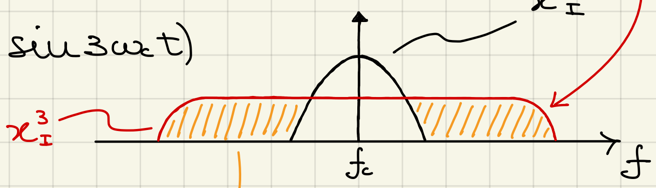
$$x(t) = x_I(t)\cos\omega_c t - x_Q(t)\sin\omega_c t$$

$$\alpha_3 x^3(t) = \alpha_3 \underline{\underline{x_I^3(t)}}\left(\frac{1}{4}\cos\omega_c t + \frac{3}{4}\underline{\underline{\cos\omega_c t}}\right) -$$

$$- \alpha_3 \underline{\underline{x_Q^3(t)}}\left(\frac{3}{4}\underline{\underline{\sin\omega_c t}} - \frac{1}{4}\sin 3\omega_c t\right)$$

signal at carrier frequency is distorted

$$y(t) = \alpha_1 x(t) + \alpha_3 x^3(t)$$

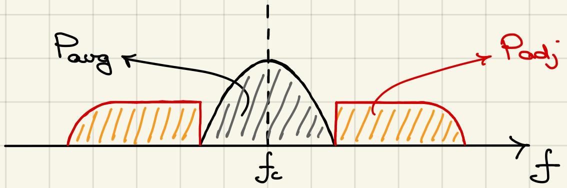


spectral regrowth ← wider bandwidth is due to the power elevation

Non-linearity degrades → EVM (inband disturbance)
→ ACPR

$$\left[\text{ACPR} := \frac{\overset{\to P_{adj}}{\text{Power leaking in adjacent channel}}}{\underset{\to P_{avg}}{\text{Power of the signal}}} \right]$$

(Adjacent Channel Power Ratio)

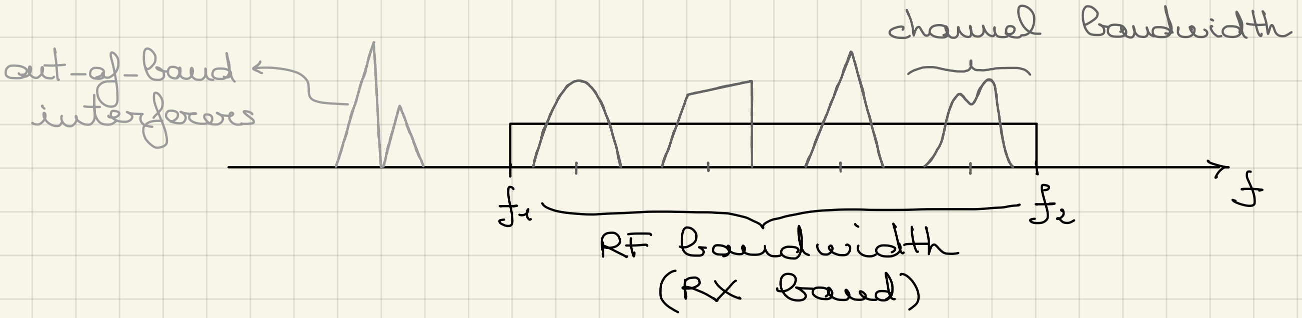


⟹ Trade-off in amplifiers between linearity and power efficiency ($\eta = \frac{P_{out}}{P_{DC}}$)

RX block diagram

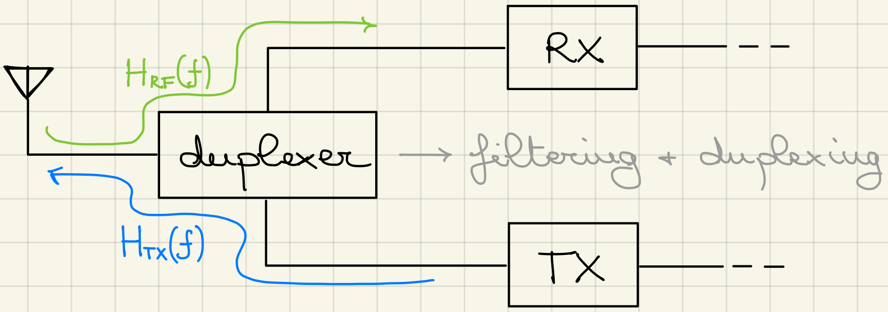
Multi-user communication system

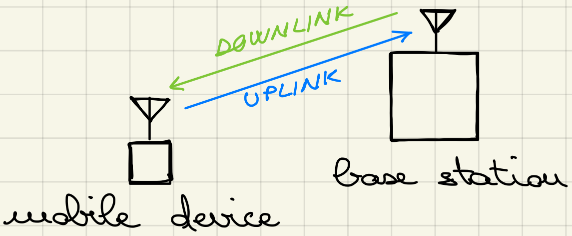
→ MULTIPLE ACCESS to the channel
e.g. FDMA (Frequency Division Multiple Access):



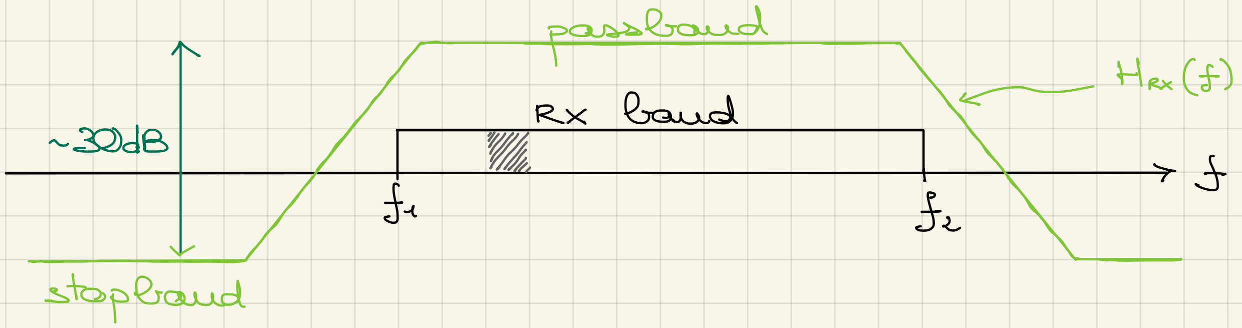
⟹ RX has to perform:

1. BAND selection (Duplexer): out-of-band rejection
2. CHANNEL selection: • cannot be performed at RF ⊛
   • tunable filters have worse performance than fixed freq. filter

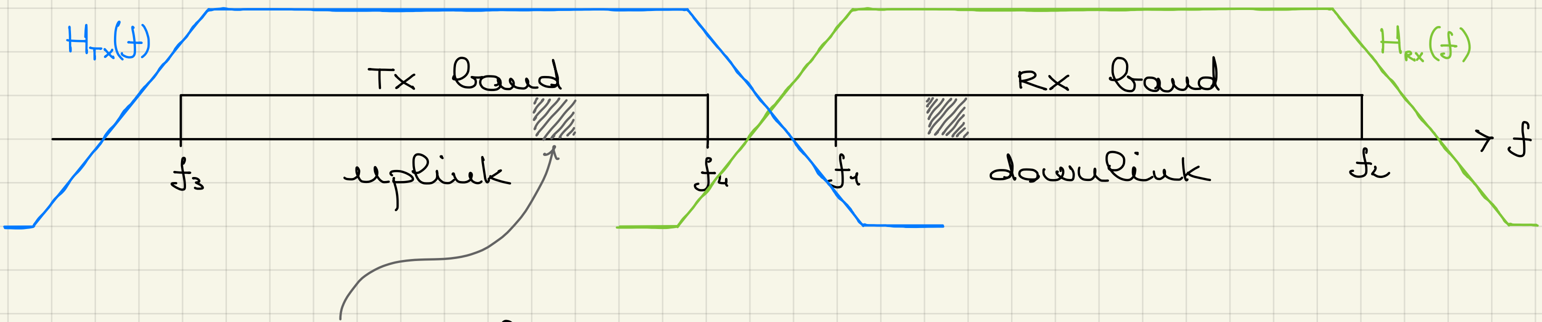




1. BAND selection



Duplexing e.g. FFD (Frequency Division Duplexing)



2. CHANNEL selection

⊛ Example: $f_{RF} = 1\,GHz = \frac{\omega_0}{2\pi}$ $f_{BW} = 200\,KHz = \frac{\Delta\omega}{2\pi}$ (≈ channel)

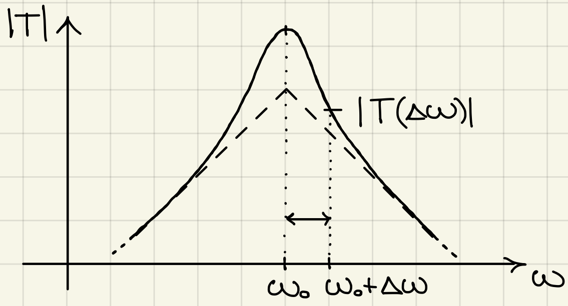
-3dB BW of LC filter

LC filter:
(2nd order)

$$|T(\Delta\omega)| \simeq \frac{\omega_0/2Q}{\Delta\omega}$$

↑

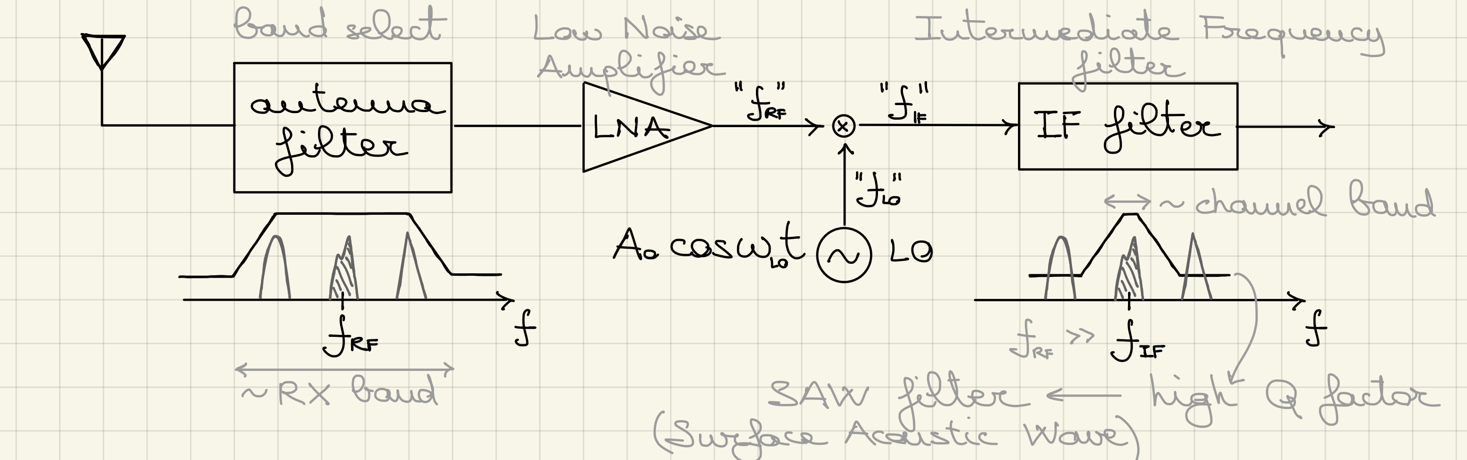
$$\frac{\Delta\omega}{\omega_0} << \frac{1}{4Q^2}$$



If $|T(\Delta\omega)| = 10^{-3}$ then $Q = 2{,}5 \cdot 10^{6}$ → too big!

Filtering a signal with very narrow bandwidth at a center frequency in the RF range would need a too high quality factor of the filter.

$Q \propto \frac{\omega_0}{\Delta\omega}$ → to reduce Q of filter must reduce center freq.

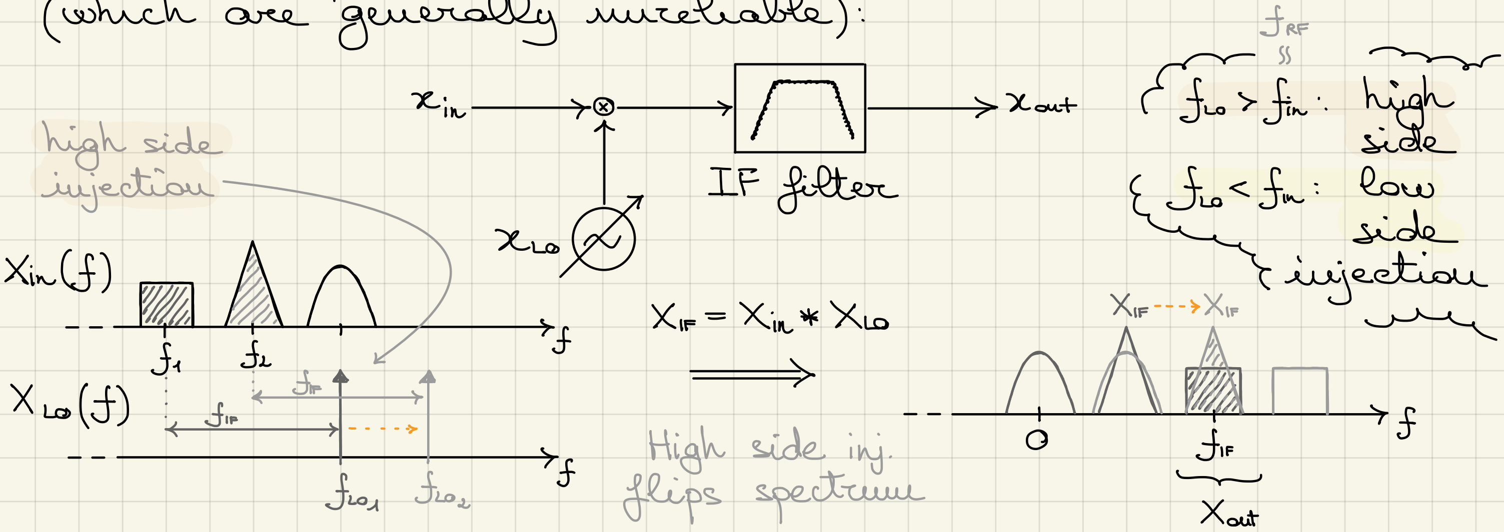


Since we need a lower center frequency the IF filter will be centered around:

$$f_{IF} = |f_{RF} - f_{LO}| \longleftarrow \cos\omega_{RF}t \cdot \cos\omega_{LO}t = \frac{1}{2}\cos\left(\omega_{RF}-\omega_{LO}\right)t + \frac{1}{2}\cos\left(\cancel{\omega_{RF}+\omega_{LO}}\right)t$$

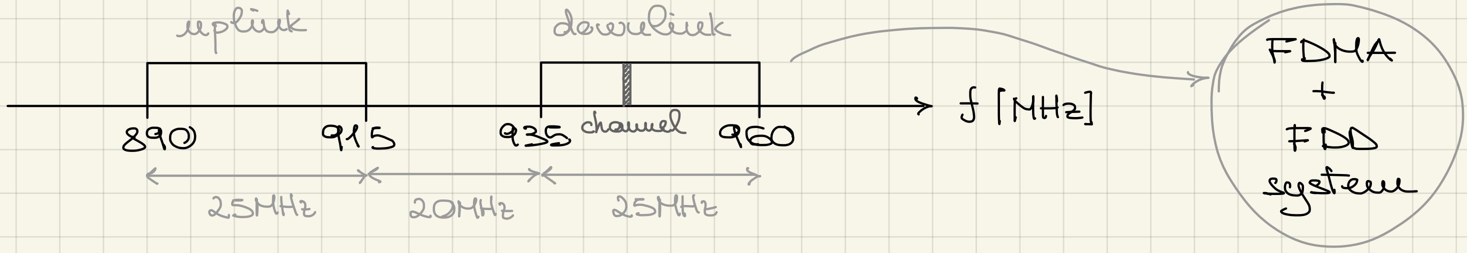
This type of channel selection at the receiver is called <u>HETERODYNE RX architecture</u>
"different" "frequency"

Such architecture also allows to filter at different central frequencies without the need of tunable filters (which are generally unreliable):



With a variable local oscillator we can shift the input spectrum and filter different channels with the same IF filter.

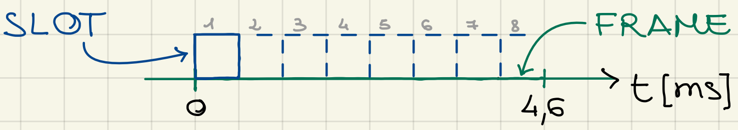
<u>Example</u>: GSM (Global System for Mobile) "2G" standard



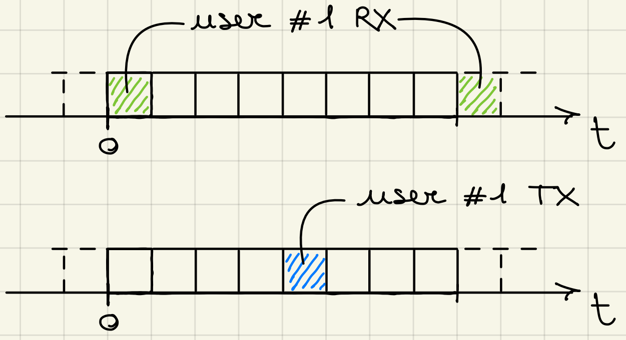
- Each band is divided into 125 carriers:

$\frac{25MHz}{125} = 200KHz$ frequency separation of channels (channel BW ~ 150KHz + guard freq. ~ 50KHz)

- Each channel is shared by 8 users:



$\frac{4,6\,ms}{8} = 0,575\,ms$ slot time

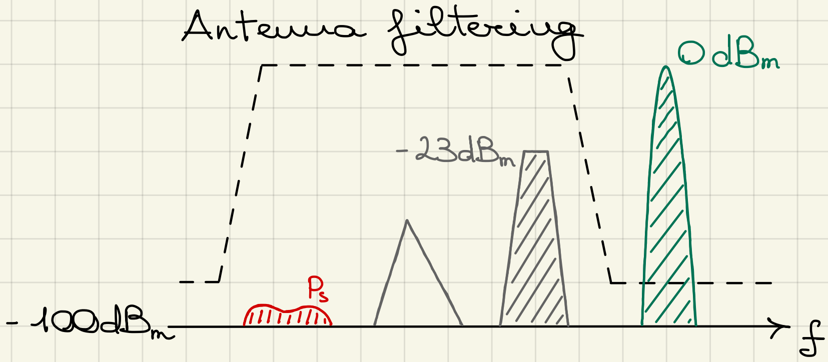


TDMA
(Time Division Multiple Access)
+
TDD
(Time Division Duplexing)

In order to use TDMA and TDD (non-continuous transmission and reception) you need digital modulation → GMSK modulation which is a CPM (Continuous Phase Modulation)

has constant envelop ←

All these specs were chosen to maximize the efficiency of mobile devices. Typical sensitivity: $-99\,dB_m$



minimum signal
||
sensitivity: $P_s = -99\,dB_m$

out-of-band interferers: $P_B = 0\,dB_m$

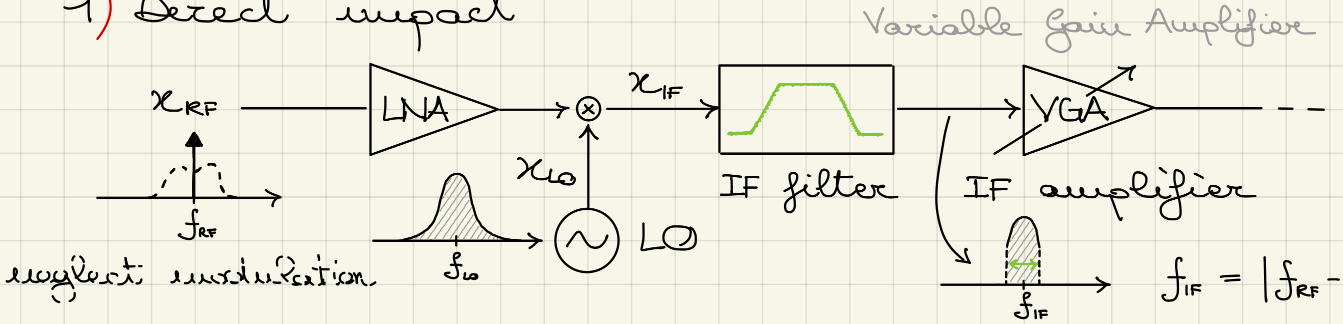
in-band interferers: $P_B = -23\,dB_m$

$$dB_m = 10\log_{10} P_{[mW]}$$

e.g.:
$0\,dB_m = 1\,mW$
$30\,dB_m = 1\,W$
$-20\,dB_m = 10\,\mu W$
$-100\,dB_m = 10^{-13}\,W$

Impact of phase noise on RX performance

1) Direct impact



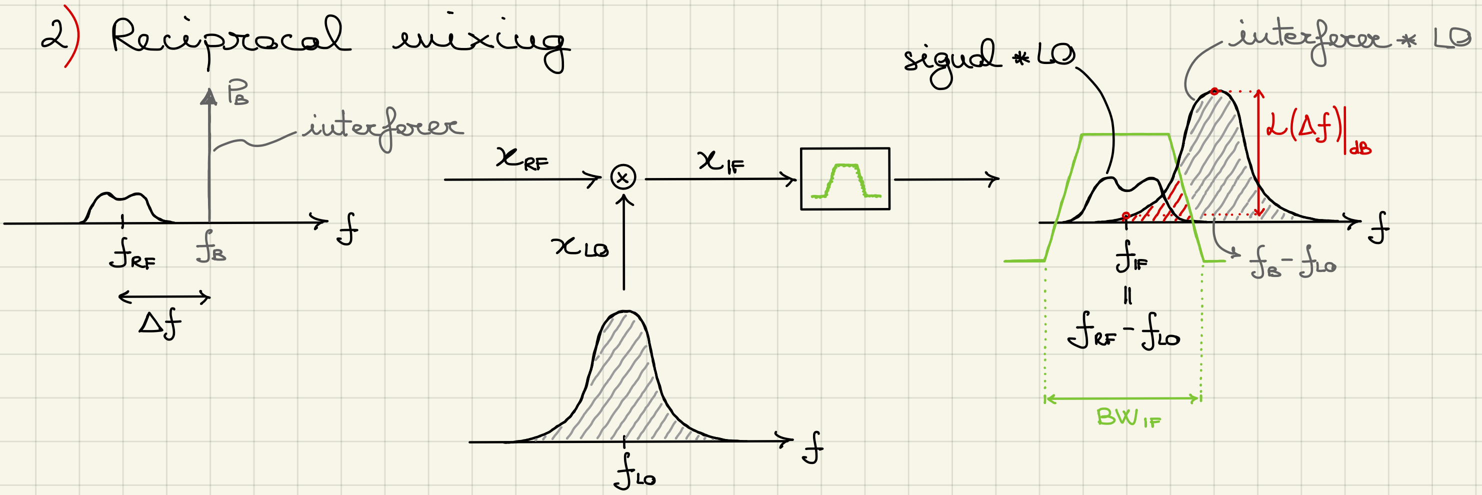
$f_{IF} = |f_{RF} - f_{LO}|$

$$A\, x_{RF}(t)\cdot x_{LO}(t) = x_{IF}(t) \Rightarrow A\, X_{RF}(f) * X_{LO}(f) = X_{IF}(f)$$

$$x_{LO}(t) = A_{LO}\cos[\omega_{LO}t + \varphi_n] \Longrightarrow SNR \leq \frac{1}{\sigma_\varphi^2}$$

The phase noise degrades the signal-to-noise ratio of the receiver (as anticipated when discussing the transmitter phase noise).

2) Reciprocal mixing



The mixing of a nearly disturbance with the noisy LO causes some additional phase noise to fall in-band.

$$\mathcal{L}(\Delta f) := \frac{S_n(f_{IF})}{P_B} \longrightarrow S_n(f_{IF}) = \mathcal{L}(\Delta f)\cdot P_B$$

$S_n(f_{IF}) \to \frac{P_n(f_{IF})}{\text{res. bw.}}$

$\mathcal{L}(\Delta f) \to dBc/Hz$

$$\Longrightarrow \boxed{SNR = \frac{P_S}{P_n(f_{IF})} \simeq \frac{P_S}{\mathcal{L}(\Delta f)P_B\cdot BW_{IF}}}$$

$P_B\int_{BW_{RF}}\mathcal{L}(f)df = P_B BW_{RF}$

assuming $S_n \equiv S_n(f_{IF})$ over entire BW_{IF}

$$\Longrightarrow \left[SNR|_{dB} = 10\log_{10}SNR = \underset{[dBm]}{P_s} - \underset{[dBm]}{P_B} - \underset{[dBc/Hz]}{\mathcal{L}(\Delta f)} - 10\log_{10}(BW_{IF})\right]$$

Example: GSM

$P_S = -99dBm$

$P_B = -40dBm$

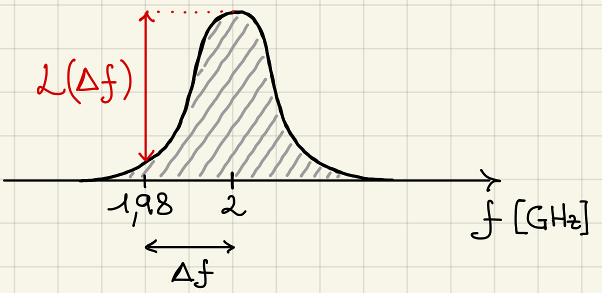
(out-of-band interferer at 0dB attenuated by an antenna filter by 40dB)

$f_{RF} = 2{,}01 GHz$ $\quad BW_{RF} = 200kHz$

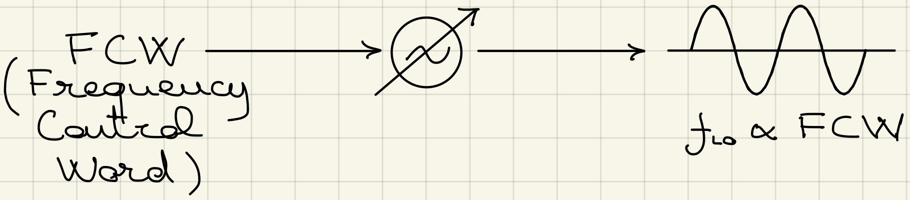
$f_B = 2{,}03 GHz$ $\quad SNR > 50dB$

$f_{LO} = 2{,}00 GHz$ $\quad \underline{\mathcal{L} = ?}$

$$\mathcal{L}(\Delta f)|_{dB} = P_s - P_B - SNR|_{dB} - 10\log_{10}BW_{RF}$$
$$= -99 + 40 - 50 - 53 =$$
$$= -162\,\frac{dBc}{Hz} \text{ at } \underset{\Delta f}{\underset{\uparrow}{20\,MHz}}$$



Frequency Synthesizers

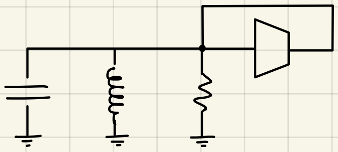


- Accuracy: $\frac{\Delta f_{LO}}{f_{LO}}$ (impaired by aging + drift)

  e.g. GSM standard requires $\frac{\Delta f}{f} \leq 0{,}1$ ppm $= 10^{-7}$

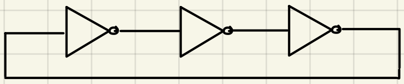
  $f = 1$ GHz $\longrightarrow$ $\Delta f \leq 100$ Hz

LC oscillator: $f \propto \frac{1}{\sqrt{LC}}$ $\quad \left|\frac{\Delta f}{f}\right| \approx \frac{1}{2}\left|\frac{\Delta L}{L}\right| + \frac{1}{2}\left|\frac{\Delta C}{C}\right|$



1÷10%

RC oscillator: $f \propto \frac{1}{RC}$ $\quad \left|\frac{\Delta f}{f}\right| \approx \left|\frac{\Delta R}{R}\right| + \left|\frac{\Delta C}{C}\right|$



- Resolution: minimum (controlled) Δf of LO
  - for channel spacing ~ 100 kHz
  - for temperature compensation ~ Hz

- Settling time: channel switching time
  - switch from one frequency to another at each frame
  - typically ~ 100 μs or even ~ 10 μs

- Spurious content: reciprocal mixing

- Phase noise

- Pulling: sensitivity of frequency to supply or load changes $\left(\frac{\Delta f}{\Delta V_{DD}}\right)$

To improve accuracy: master/slave approach

$\overset{\text{slave}}{\text{RC/LC oscillators}} \leftarrow$ Crystal oscillators (Quartz) $\leftarrow \overset{\text{master}}{\text{Atomic clocks}}$

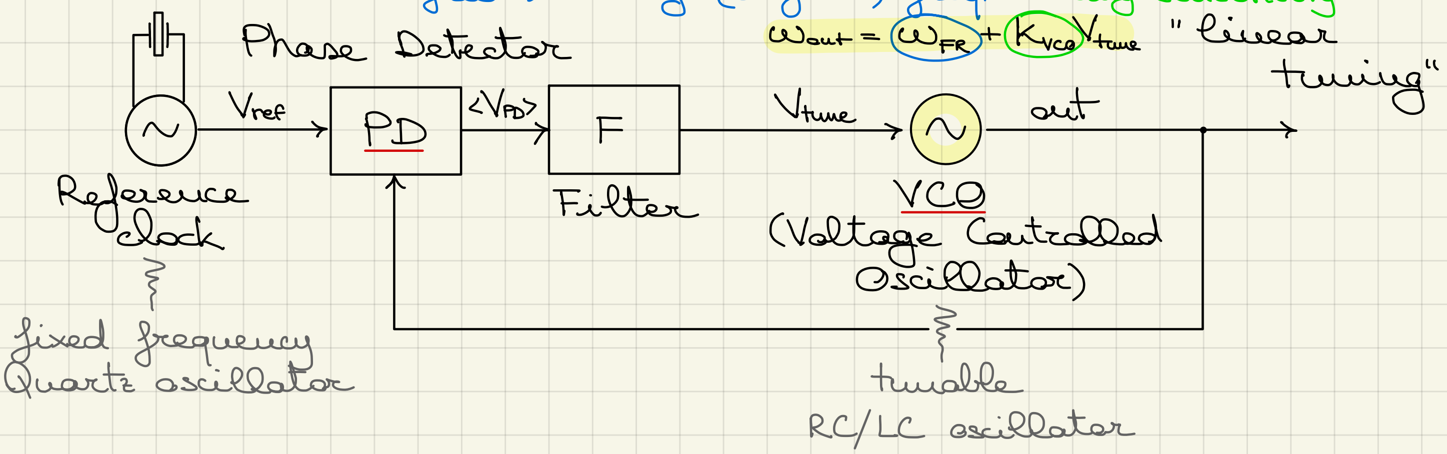
RC/LC	Quartz	Atomic
× poor accuracy	✓ good accuracy accuracy ≈ 100 ppm aging ≈ 0,5 ppm/year drift ≈ 0,5 ppm in 0-75°C	✓ best accuracy aging ≈ 10^{-9} s/day
✓ tunable	× not tunable	
✓ can operate at large frequency	× low-frequency	

e.g. TCXO (Temperature Compensated Crystal Oscillator)

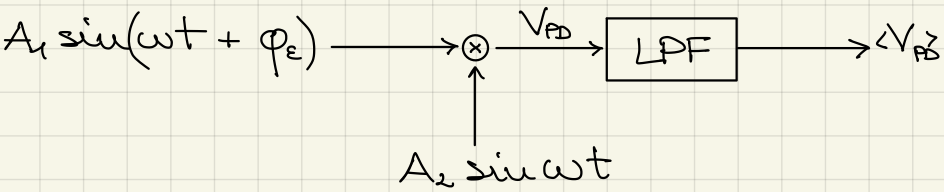
We will focus on the "slaves" (RC/LC and Quartz oscillators)

Phase - Locked Loop (PLL)

$$\omega_{out} = \overset{\text{free-running (angular) freq.}}{\omega_{FR}} + \overset{\text{tuning sensitivity}}{K_{VCO}} V_{tune}$$ "linear tuning"



Phase Detector

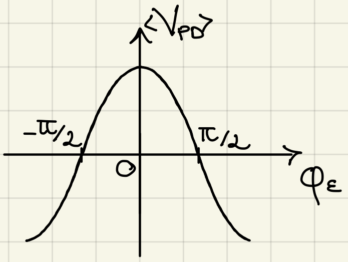


(this is just one type of PD)

$$V_{PD} = -\frac{A_1A_2}{2}\underbrace{\cos(2\omega t + \varphi_\varepsilon)}_{\text{fast}} + \frac{A_1A_2}{2}\underbrace{\cos(\varphi_\varepsilon)}_{\text{DC}}$$

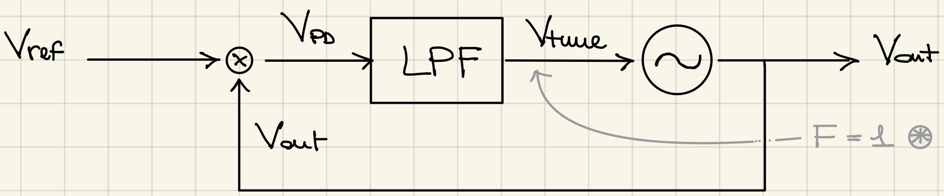
$$\langle V_{PD} \rangle \simeq \frac{A_1A_2}{2}\cos\varphi_\varepsilon \quad \text{if} \quad BW_{LPF} \ll 2\omega$$

Static PD characteristic:



Notation change

$$V_{ref} = A_r \sin\phi_{ref}$$



$$\underset{\text{absolute phase}}{\phi_{ref}} = \omega_{ref}t + \underset{\text{excess phase}}{\varphi_{ref}}$$

$$V_{out} = A_o\cos\phi_{out} \Longrightarrow \overset{V_{tune}}{\overset{\wr\wr}{\langle V_{PD}\rangle}} \simeq \overbrace{\frac{A_r\cdot A_o}{2}}^{K_{PD}:\ \text{PD gain}}\sin\overbrace{(\phi_{ref}-\phi_{out})}^{\phi_\varepsilon:\ \text{phase error}} = K_{PD}\sin\phi_\varepsilon$$

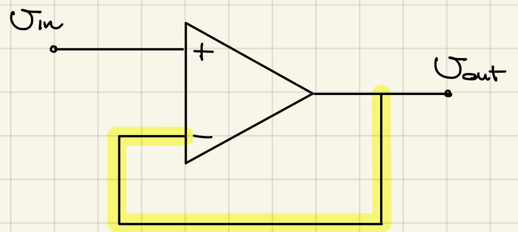
We can now compute the effects of the loop on the phase error, which indicates how well the output follows the reference:

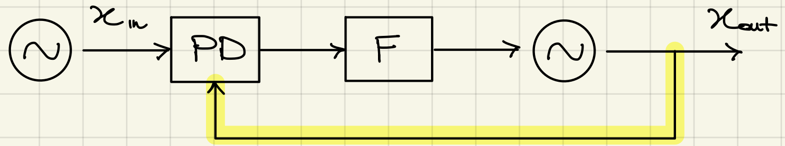
$$\frac{d\phi_\varepsilon}{dt} = \dot{\phi}_\varepsilon = \dot{\phi}_{ref} - \dot{\phi}_{out} = \omega_{ref} - (\omega_{FR} + K_{VCO}\cdot V_{tune}) =$$

$$= \underbrace{\omega_{ref} - \omega_{FR}}_{\Delta\omega\ [rad/s]} - \underbrace{K_{VCO}\cdot K_{PD}}_{K\ \left[\frac{rad/s\cdot\cancel{V}}{\cancel{V}}\right]}\sin\phi_\varepsilon$$

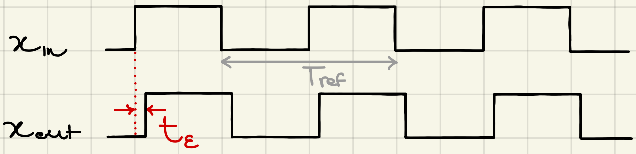
$$\Longrightarrow \boxed{\dot{\phi}_\varepsilon = \Delta\omega - K\sin\phi_\varepsilon}$$

First-order diff. equation → first-order⊛ PLL





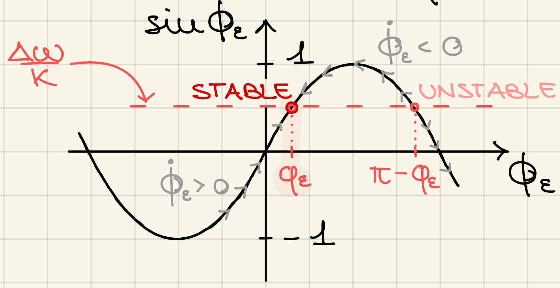
Voltage follower ← - - → Phase follower



$$\omega_{ref} = \frac{2\pi}{T_{ref}} \qquad \varphi_\varepsilon = \omega_{ref}\cdot t_\varepsilon$$

$$\dot{\phi}_\varepsilon = \Delta\omega - K\sin\phi_\varepsilon \qquad \phi_\varepsilon(t) \text{ unknown}$$

Equilibrium points: $\dot{\phi}_\varepsilon = 0 \Longrightarrow \sin\phi_\varepsilon = \frac{\Delta\omega}{K}$



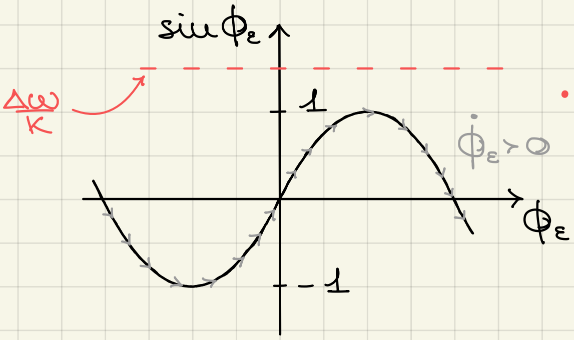
- If $\left|\frac{\Delta\omega}{K}\right| < 1$ the system has 2 equilibrium points

$$\dot{\phi}_\varepsilon < 0 \Longleftrightarrow \Delta\omega - K\sin\phi_\varepsilon < 0$$

ϕ_ε decreasing $\qquad \sin\phi_\varepsilon > \frac{\Delta\omega}{K}$

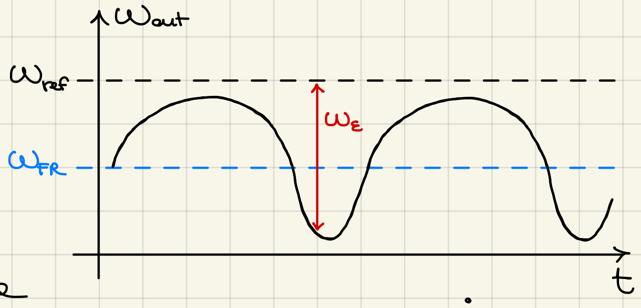
$$\dot{\phi}_\varepsilon > 0 \iff \sin\phi_\varepsilon < \frac{\Delta\omega}{K}$$

ϕ_ε increasing



- If $\left|\frac{\Delta\omega}{K}\right| > 1$ the system has no equilibrium points

ϕ_ε is always increasing or decreasing.



$$\omega_{out} = \omega_{FR} + K\sin\phi_\varepsilon(t)$$

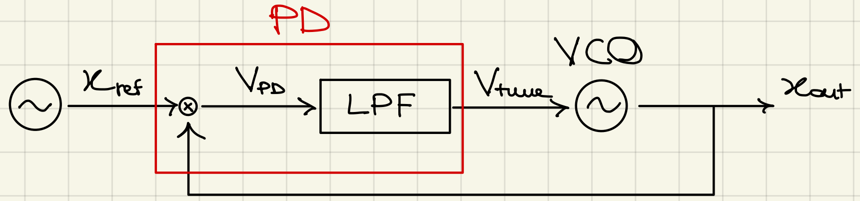
$$\omega_\varepsilon = \dot\phi_\varepsilon = \omega_{ref} - \omega_{FR}$$

The distorted sinusoid is due to the fact that as ω_{out} approaches ω_{ref}, $\omega_\varepsilon = \dot\phi_\varepsilon$ decreases, therefore ω_{out} which depends on $\phi_\varepsilon(t)$ varies slowlier.

Conclusion: if $\left|\frac{\Delta\omega}{K}\right| < 1$ then $\phi_\varepsilon(t) \longrightarrow \left[\varphi_\varepsilon = \arcsin\left(\frac{\Delta\omega}{K}\right)\right]$ (stable equilibrium point)

↳ steady-state phase error depends on the freq. offset between reference and free-running freq. of the VCO

To summarize:



$$\begin{cases} x_{ref} = A_r \sin\phi_{ref} \\ x_{out} = A_o \cos\phi_{out} \end{cases}$$

- PD: multiplier + ideal LPF $\langle V_{PD}\rangle = K_{PD}\sin\phi_\varepsilon$
- VCO: linear tuning $\omega_{out} = \omega_{FR} + K_{VCO} V_{tune}$

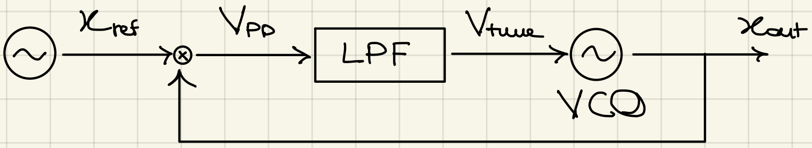
$\phi_\varepsilon := \phi_{ref} - \phi_{out}$ $\Delta\omega := \omega_{ref} - \omega_{FR}$ $K := K_{PD}\cdot K_{VCO}\ \left[\frac{rad}{s}\right]$

$$\dot\phi_\varepsilon = \Delta\omega - K\sin\phi_\varepsilon$$

- $\left|\frac{\Delta\omega}{K}\right| < 1$: $\varphi_\varepsilon = \arcsin\left(\frac{\Delta\omega}{K}\right)$ equilib. point "LOCK STATE"
- $\left|\frac{\Delta\omega}{K}\right| > 1$: no equilib. points "OUT-OF-LOCK"

"LOCK RANGE" $\Delta\omega_L = K$

Interpretation:



IMPOSE LOCK

impose equality at steady-state

$$\omega_{out} = \omega_{FR} + K_{VCO} V_{tune} = \omega_{ref}$$

$$\Longrightarrow \quad V_{tune} = \frac{\omega_{ref} - \omega_{FR}}{K_{VCO}} = \frac{\Delta\omega}{K_{VCO}}$$

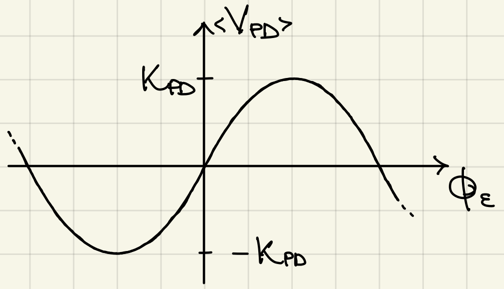
$$V_{tune} = \langle V_{PD} \rangle = K_{PD} \cdot \sin\phi_\varepsilon = \frac{\Delta\omega}{K_{VCO}} \longrightarrow \sin\phi_\varepsilon = \frac{\Delta\omega}{K_{VCO}K_{PD}} \rightarrow K$$

$$\Longrightarrow \quad \boxed{\sin\phi_\varepsilon = \frac{\Delta\omega}{K}}$$

same result of diff. equation approach

The lock state condition and lock range can be intuitively explained by considering that the PD output is limited:

$$\langle V_{PD} \rangle_{max} = K_{PD} > \frac{\Delta\omega}{K_{VCO}} \quad \text{to reach lock}$$

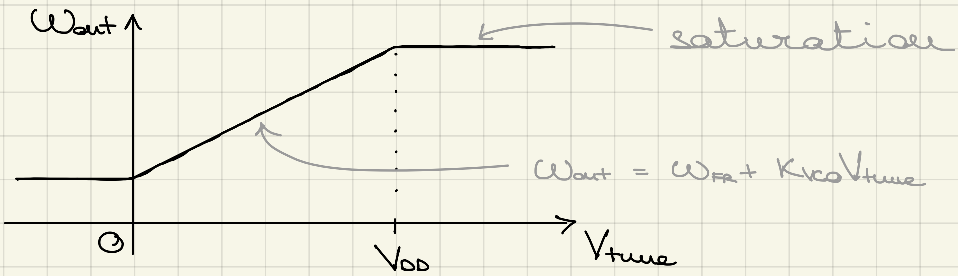


$$\Longrightarrow \quad \boxed{\frac{\Delta\omega}{K_{VCO}K_{PD}} < 1}$$

same condition of diff. eq. approach

The limited dynamic range of PD limits lock range.

Of course, the VCO also limits locks range:



Perturbation analysis of the differential equation based on linearization:

$$\dot{\phi}_\varepsilon = \Delta\omega - K \sin\phi_\varepsilon$$

Hp: • $\left|\frac{\Delta\omega}{K}\right| < 1$ stable equilib. exists

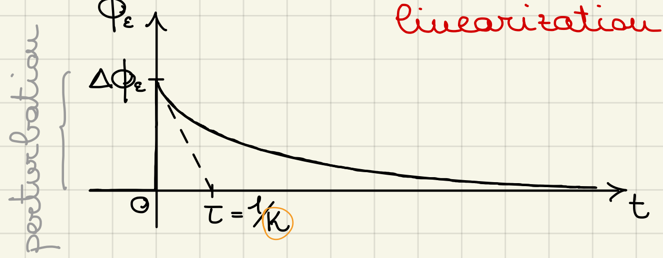
• $\Delta\phi_\varepsilon \ll 1\,\text{rad}$ small perturbation

If $\Delta\omega = 0$: $\dot{\phi}_\varepsilon = -K \sin\phi_\varepsilon$

$$\xrightarrow{\text{linearization}} \quad \dot{\phi}_\varepsilon = -K\phi_\varepsilon \Longrightarrow \phi_\varepsilon(t) = \Delta\phi_\varepsilon \cdot e^{-Kt}$$

or in Laplace domain

$$s\Phi_\varepsilon = -K\Phi_\varepsilon \Longrightarrow \text{pole at } s = -K$$



Let's compute the input-to-output transfer function Φ_{out} vs. Φ_{ref} of the PLL:

$$\omega_{out} = \omega_{fr} + K_{VCO} V_{tune}(t) =$$
$$= \underbrace{\omega_{fr} + K_{VCO} V_{tune,0}}_{\omega_{out,0}} + K_{VCO} v_{tune}(t) =$$
$$= \omega_{out,0} + \overline{K_{VCO} v_{tune}(t)} =$$
$$= \omega_{out,0} + K_{VCO} K_{PD} [\phi_{ref}(t) - \phi_{out}(t)]$$

$$\Phi_{out} = \int_{-\infty}^{t} \omega_{out}(t')\,dt' = \cancel{\omega_{out,0} \cdot t} + \varphi_{out}(t)$$

Φ_{out}: ABSOLUTE PHASE

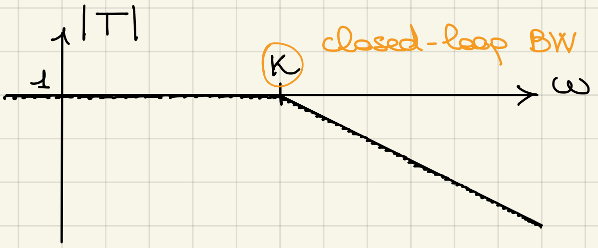
$\varphi_{out}(t)$: EXCESS PHASE

consider system only in its "small signal" variations

$$\Longrightarrow \dot{\varphi}_{out} = K_{VCO} \cdot K_{PD} [\varphi_{ref}(t) - \varphi_{out}(t)] = \dot{\Phi}_{out} - \omega_{out,0}$$ neglect DC component

$$s\Phi_{out}(s) = K [\Phi_{ref}(s) - \Phi_{out}(s)]$$

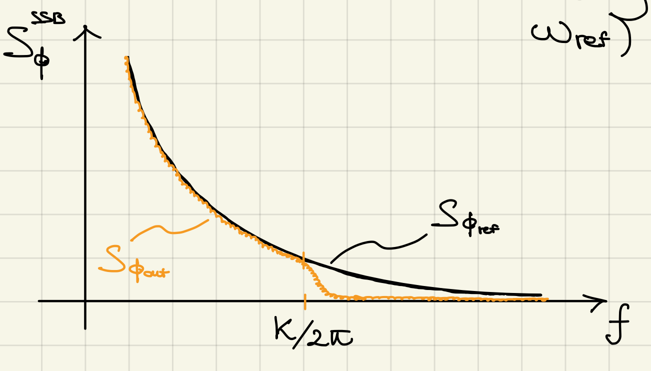
$$\Phi_{out}/\Phi_{ref} = \boxed{K/s+K = T(s)}$$



$$\boxed{T(s) = \frac{\Phi_{out}}{\Phi_{ref}} = \frac{s\,\Phi_{out}}{s\,\Phi_{ref}} = \frac{\Omega_{out}}{\Omega_{ref}}}$$

where $\Phi = \mathcal{L}[\varphi]$ and $\Omega = \mathcal{L}[\omega]$ (Lagrange transform)

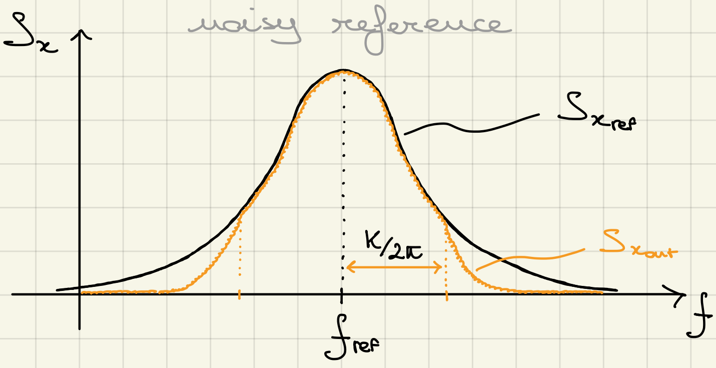
Interpretation: in this PLL, the VCO "follows" the phase and frequency of the reference clock with BW = K.
Only slow variations of ϕ_{ref} (or ω_{ref}) are followed by the VCO.



$$S_{\phi out} = |T(f)|^2 \cdot S_{\phi ref}$$

Low-pass filtering of input phase noise

$$\mathcal{L}(f) \simeq \frac{S_\phi^{SSB}(f)}{2}$$



Band-pass filtering of input signal

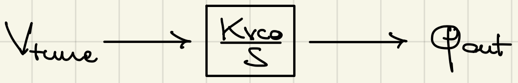
Note: trade-off between BW and LOCK RANGE

Equivalent model of linear PLL

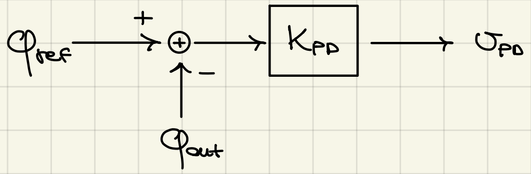
- VCO: $\varphi_{out}(t) = \int_{-\infty}^{t} K_{VCO}\cdot v_{tune}(t')\,dt' =$

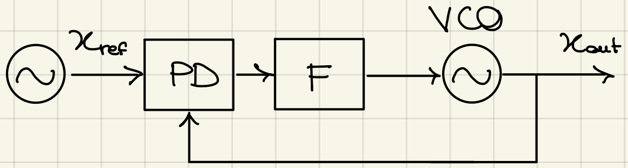
excess phase only

$$\Phi_{out}(s) = \frac{K_{VCO}}{s}\cdot V_{tune}(s)$$

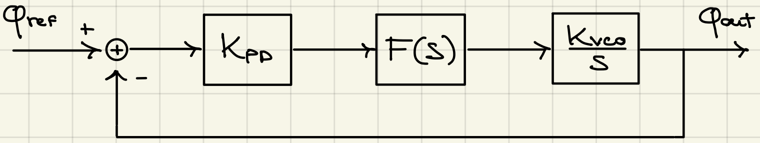


- PD: $v_{PD} = K_{PD}\left[\overbrace{\phi_{ref}-\phi_{out}}^{\phi_e}\right]$ linear PD





PLL



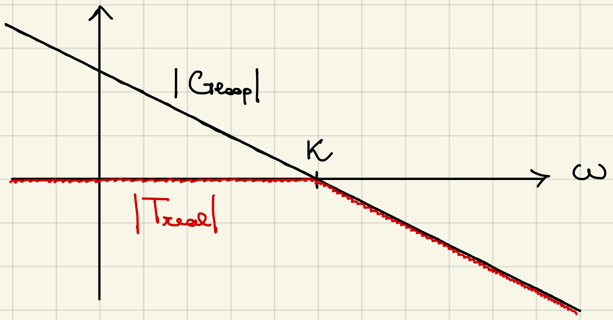
linear continuous-time (average) model of a PLL

First order PLL: $F(s) = 1$

(neglecting signs)

$$G_{loop}(s) = -K_{PD}\cdot F(s)\cdot\frac{K_{VCO}}{S} = -\frac{K}{S}$$

$$T_{ideal}(s) = 1$$

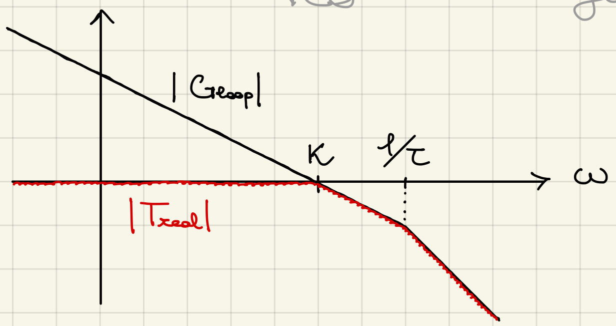


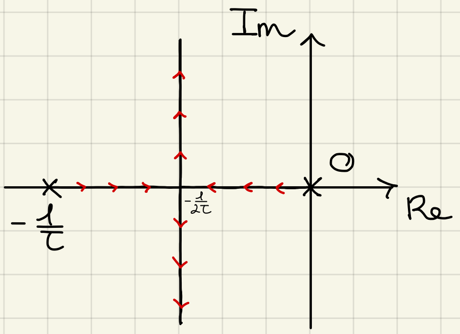
Second order PLL: $F(s) = \frac{1}{1+s\tau}$

(more realistic than 1st order)

$$G_{loop}(s) = -\underbrace{\frac{K}{s}}_{VCO}\,\underbrace{\frac{1}{1+s\tau}}_{filter}$$

$$T_{ideal}(s) = 1$$

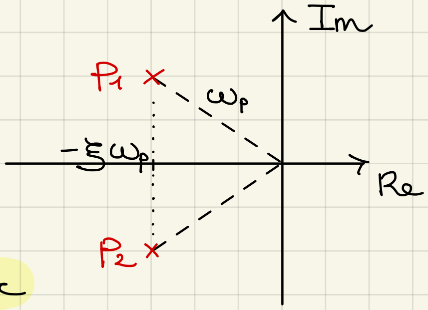




$$T(s) = \frac{G_{loop}(s)}{1+G_{loop}(s)} = \frac{K/s\,\frac{1}{1+s\tau}}{1+K/s\,\frac{1}{1+s\tau}} = \frac{K}{s^2\tau+s+K} =$$

$\overset{\|}{T_{real}}$

$$= \frac{1}{s^2\frac{\tau}{K}+\frac{s}{K}+1} = \frac{1}{\frac{s^2}{\omega_p^2}+\frac{2\xi s}{\omega_p}+1}$$



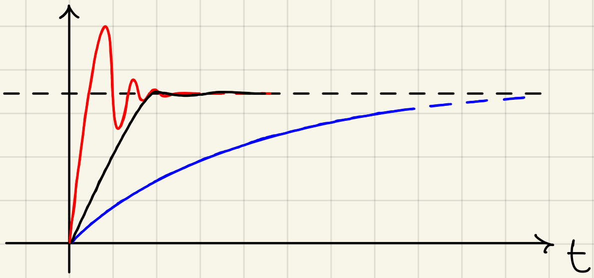
$\omega_p = \sqrt{\frac{K}{\tau}}$ natural frequency

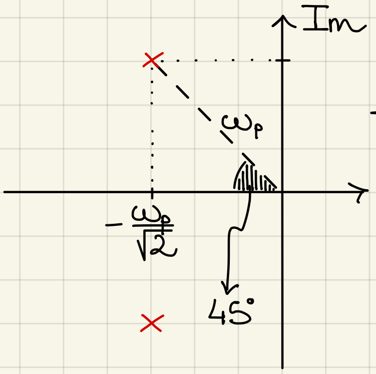
$\xi = \frac{1}{2\sqrt{K\tau}} = -\frac{Re(p)}{|p|}$ damping factor

$$p_{1\text{-}2} = -\xi\omega_p \pm j\sqrt{1-\xi^2}\,\omega_p$$

By choice of τ and K you can set ω_p and ξ.

Closed loop poles at 45° in Gauss plane (Best trade-off between overshoot and rise time):





$\xi = \cos\vartheta = \frac{1}{\sqrt{2}}$

$$\xi = \frac{1}{2\sqrt{K\tau}} = \frac{\sqrt{2}}{2} \longrightarrow K\tau = \frac{1}{2}$$

$$K = \frac{1/\tau}{2}$$

factor 2
log distance

$\Longrightarrow$ Crossover of G_{loop} (K) one octave before the second pole (1/τ)

$$\omega_p = \sqrt{\frac{K}{\tau}} = \sqrt{\frac{1}{2\tau^2}} = \frac{1}{\sqrt{2}}\cdot\frac{1}{\tau} = \sqrt{2}\,K$$

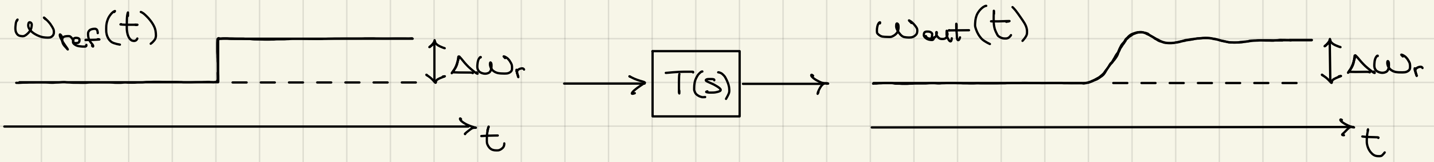
The bandwidth is not equal to K (like one would expect from the graphical approximation) but it is actually equal to $\sqrt{2}K$.

The phase margin is 63°

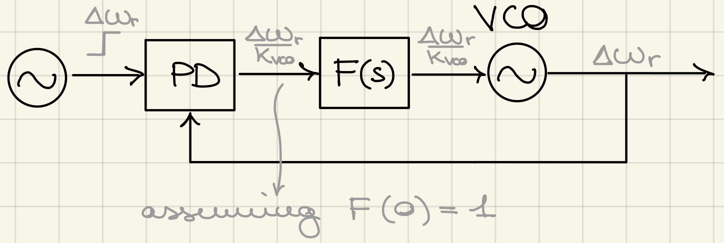
again trade-off between BW and LOCK RANGE

Static Phase Error

It is the residual error at steady-state between Φ_{out} and Φ_{ref}.

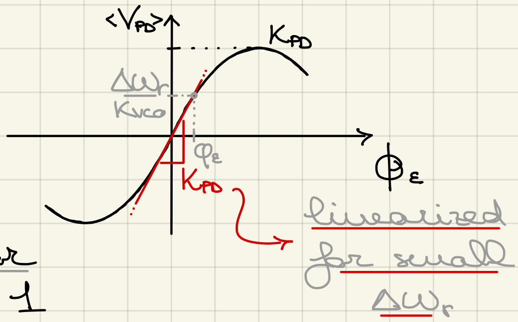


1. What is the value of ϕ_ε at steady-state?



variations at steady-state

$$\left[\ \varphi_\varepsilon \simeq \frac{\Delta\omega_r / K_{VCO}}{K_{PD}}\ \right] \Longleftarrow$$

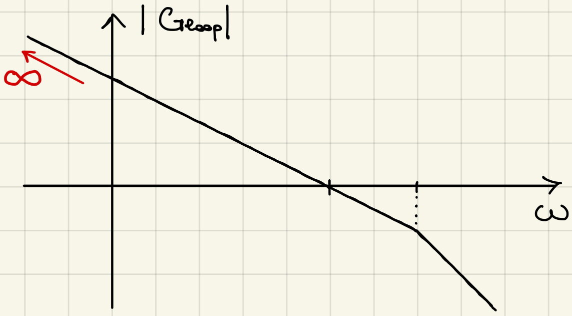


Same result obtained with diff. equations, but it holds for any order PLL that has F(0) = 1

2. Why is the static ϕ_ε not null, although $|G_{loop}| \longrightarrow \infty$ at DC?

$$|G_{loop}| = \frac{K}{s} \cdot \frac{1}{1+s\tau}$$

(2nd order PLL)

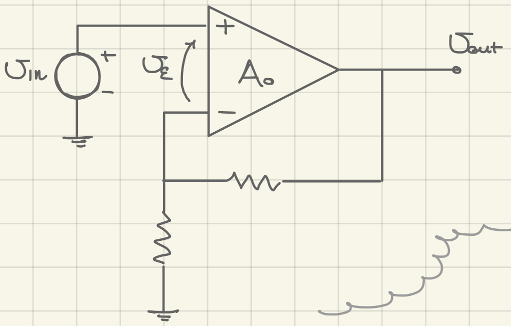


In a voltage amplifier:

$$v_\varepsilon = \frac{v_{out}}{A_0} \longrightarrow 0$$

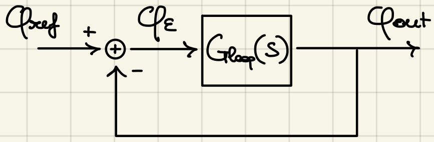
finite

∞



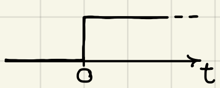
Final Value Theorem: $\left[\ \lim_{t\to\infty} \varphi_\varepsilon(t) = \lim_{s\to 0} s\,\Phi_\varepsilon(s)\ \right]$

$$\frac{\Phi_\varepsilon(s)}{\Phi_{ref}(s)} = 1 - T(s) = \frac{1}{1+G_{loop}(s)} = \frac{s(1+s\tau)}{s(1+s\tau)+K}$$

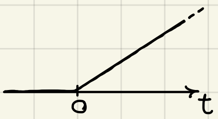


$\omega_{ref}(t) = \Delta\omega_r \cdot u(t)$, step function $u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$

$\longrightarrow \Omega_{ref}(s) = \frac{\Delta\omega_r}{s}$



$\longrightarrow \varphi_{ref}(s) = \frac{\Omega_{ref}(s)}{s} = \frac{\Delta\omega_r}{s^2}$



$$\varphi_e(s) = \frac{\Delta\omega_r}{s^2} \cdot \frac{s(1+s\tau)}{s(1+s\tau)+K}$$

$$\Longrightarrow \lim_{s\to 0} s\,\varphi_e(s) = \lim_{s\to 0} \cancel{s} \cdot \frac{\Delta\omega_r}{\cancel{s^2}} \cdot \frac{\cancel{s}(1+s\tau)}{s(1+s\tau)+K} = \frac{\Delta\omega_r}{K} \overset{\text{F.V.T.}}{=} \varphi_e(t^\infty)$$

static phase error

So the static phase error is due to the nature of the perturbation we are applying to the system. A step in frequency is actually a ramp in phase. Since the "type" (i.e. number of poles in the origin) of the transfer function $\varphi_{ref} \to \varphi_e$ is 1 in this case, the F.V.T. returns a non-null value of the static phase error.
Note that in the voltage amplifier example we were (implicitly) applying a voltage step (not a ramp) at the input so the error was (ideally) nil.

In case of a phase step. $\varphi_{ref} = \frac{\Delta\Phi}{s}$

$$\Longrightarrow \lim_{s\to 0} s\,\varphi_e(s) = \cancel{s} \cdot \frac{\Delta\omega_r}{\cancel{s}} \cdot \frac{s(1+s\tau)}{s(1+s\tau)+K} = 0 \longrightarrow \text{static phase error is nil}$$

So how can we build a PLL with zero static φ_e even after a frequency step?

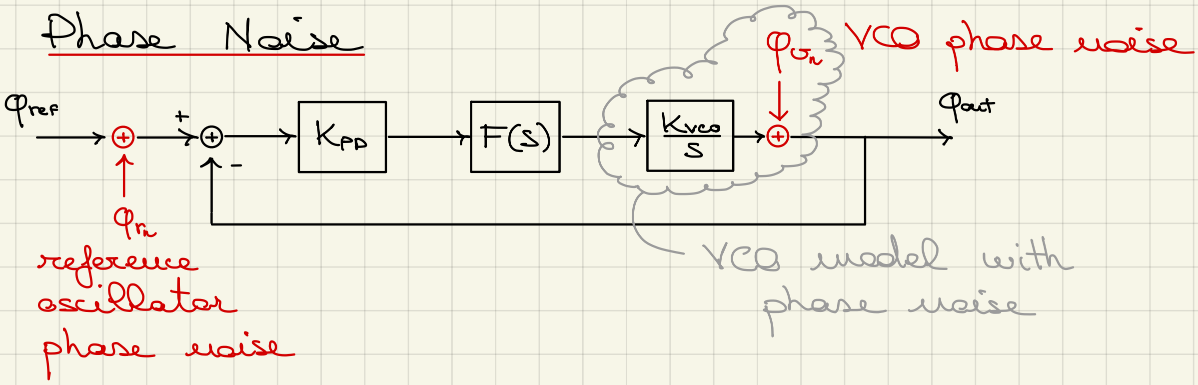
In general: G_{loop} has n integrators and φ_{ref} is of order m

$$G_{loop}(s) = \frac{K}{s^n} \cdot \frac{1}{H(s)} \qquad \varphi_{ref}(s) = \frac{\Delta}{s^m}$$

$$\lim_{t\to\infty} \varphi_e(t) = \lim_{s\to 0} s \cdot \frac{\Delta}{s^m} \cdot \frac{s^n H(s)}{s^n H(s)+K} = \lim_{s\to 0} \frac{\Delta}{K} s^{n-m+1} = \begin{cases} \frac{\Delta}{K} & n = m-1 \\ 0 & n \geq m \end{cases}$$

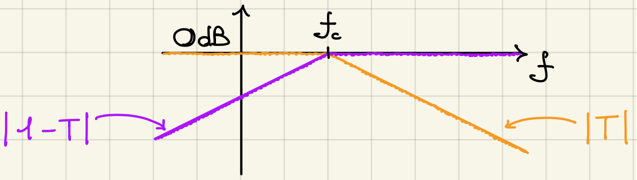
$\Longrightarrow$ Static (phase) error is zero IF the number of integrators in $G_{loop}(s)$ (= type of $G_{loop}(s)$) is at least equal to the order of the input perturbation.

Phase Noise



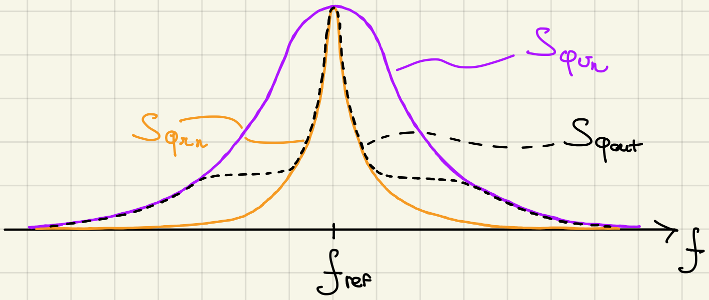
$$S_{\varphi_{out}}(f) = S_{\varphi_{rn}} |T(f)|^2 + S_{\varphi_{vn}} |1-T(f)|^2$$

$\varphi_{vn} \longrightarrow \varphi_{out}$



Interpretation:

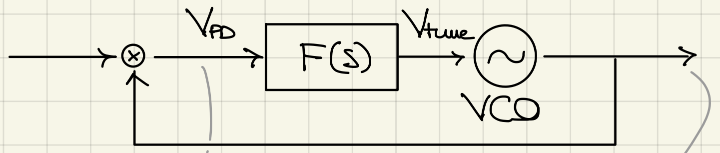
- within PLL BW the VCO follows the phase noise of the reference clock
- out of PLL BW, the VCO follows its own phase noise

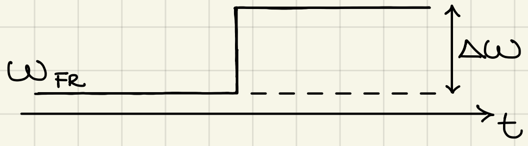


Capture Range

Consider a perturbation of the reference frequency:

$\cos(\underbrace{\omega_{FR} + \Delta\omega}_{\omega_{ref}})t$





$$\sin[\omega_{FR}t + \underbrace{\int_{-\infty}^{t} K_{vco} V_{tune}(t')dt'}]$$

for a fast step we can initially neglect this term

$$K_{PD} \sin[\Delta\omega t + \dots]$$

$$\Longrightarrow V_{tune} \simeq K_{PD} |F(\Delta\omega)| \cdot \sin[\Delta\omega t + \dots]$$

$$\Longrightarrow \quad \underset{\left|\frac{\Delta\omega}{K_{VCO}}\right|}{\underset{\parallel}{|V_{tune}|}} \leq K_{PD} \cdot |F(\Delta\omega)| \quad \text{since} \quad |\sin[..]| \leq 1$$

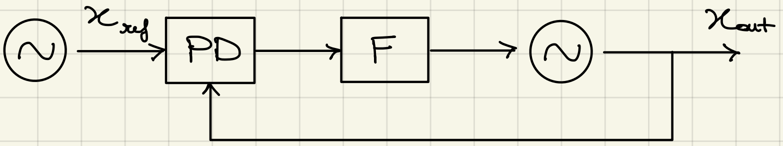
$$\left|\frac{\Delta\omega}{K_{VCO}}\right| \leq K_{PD} \cdot |F(\Delta\omega)|$$

"CAPTURE (or HOLD) RANGE"

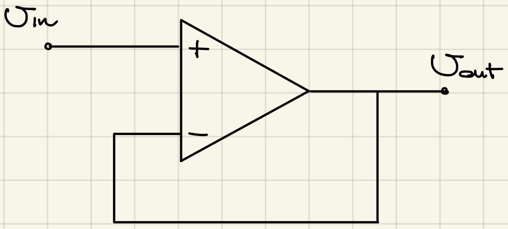
$$|\Delta\omega| \leq K_{VCO} K_{PD} \, |F(\Delta\omega)| \Longrightarrow \Delta\omega_c = K \, |F(\Delta\omega_c)|$$

The capture range indicates if the PLL can follow a quick and wide variation of the reference frequency until steady-state is reached.
The lock range indicates instead if the PLL can follow a fixed frequency already at steady state.

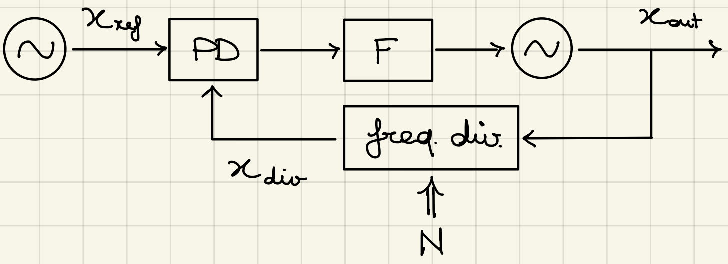
Integer-N PLL



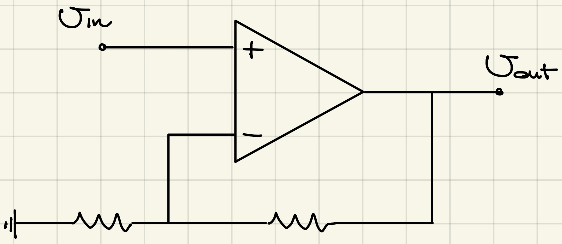
Frequency follower



Voltage follower

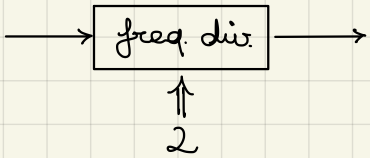


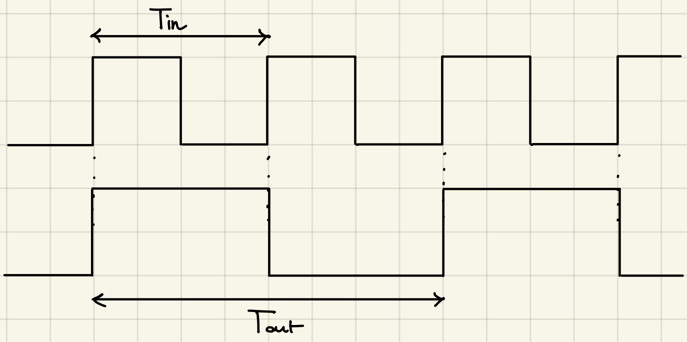
Frequency multiplier



Voltage amplifier

e.g.: Frequency divider by N = 2

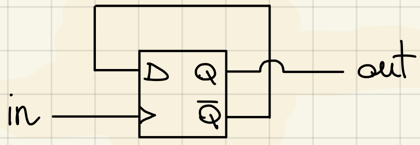




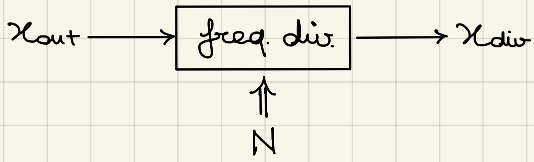
$T_{out} = 2 T_{in}$

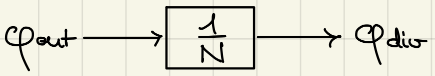
can be implemented with:

modulo - 2 counter (MSB output)



- Equivalent model of the frequency divider

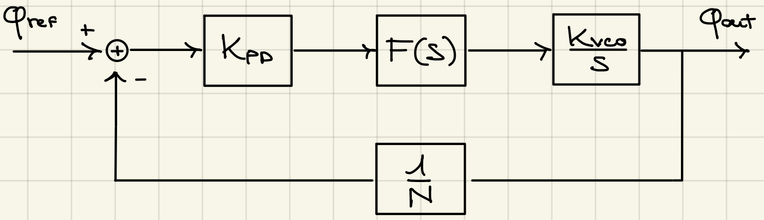




$$\varphi_{out} = \frac{\varphi_{div}}{N}$$

$$\omega_{div} = \frac{\omega_{out}}{N}$$

Equivalent model of integer-N PLL



$$G_{loop}(s) = -K_{PD}\, F(s)\, \frac{K_{vco}}{s}\, \frac{1}{N} \qquad T_{ideal}(s) = N$$

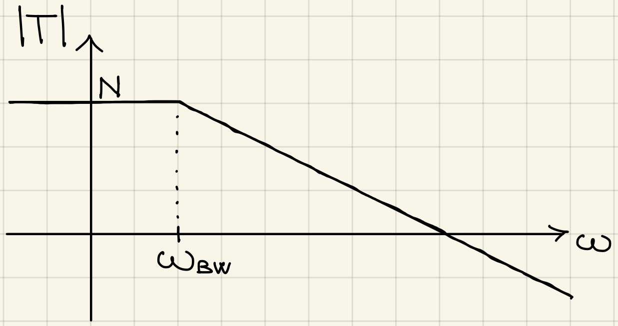
$$T(s) = \frac{N \cdot G_{loop}(s)}{1 + G_{loop}(s)}$$

Phase noise:

$$S_{\varphi out} = S_{\varphi rn} \left|T\right|^2 + S_{\varphi vn} \left|\frac{1}{1+G_{loop}}\right|^2$$

(LPF) N^2 within BW

(HPF) 1 outside BW

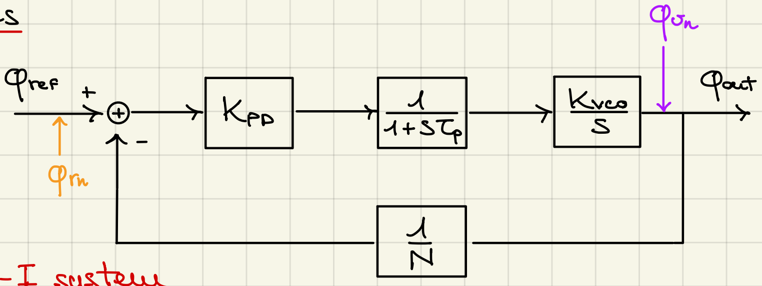


$\Longrightarrow$ Int-N PLLs amplify the reference phase noise

Type-II PLL

We introduce type-II PLLs to deal with the static phase error, as well as other issues, of type-I PLLs.

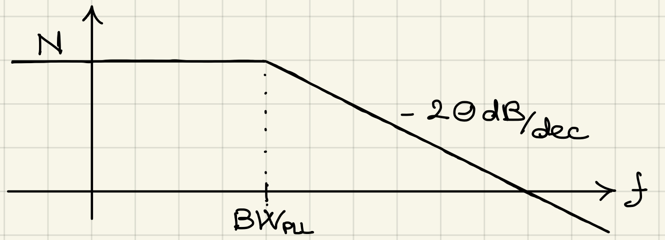
Issues of type-I PLLs



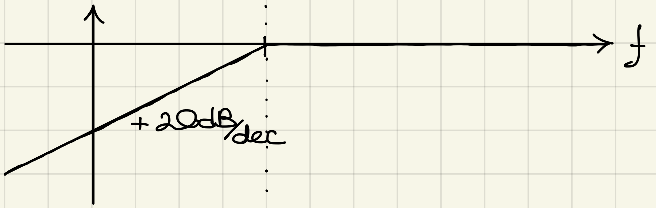
$$G_{loop}(s) = -K_{PD}\, \frac{1}{1+s\tau_p}\, \frac{K_{vco}}{\underbrace{s}_{\text{type-I system}}}\, \frac{1}{N}$$

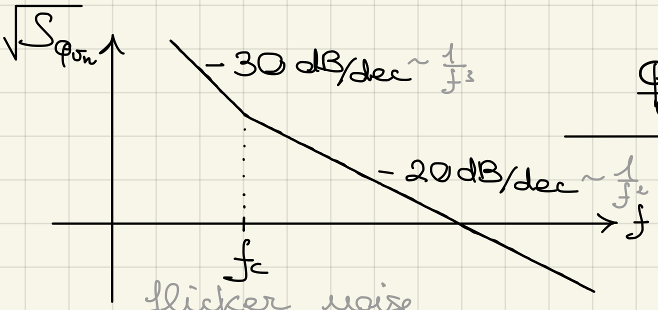
1) Limited VCO noise filtering

- $\frac{\varphi_{out}(s)}{\varphi_{rn}(s)} = N\,T(s) = N\,\frac{G_{loop}(s)}{1+G_{loop}(s)}$



- $\frac{\varphi_{out}(s)}{\varphi_{vn}(s)} = 1 - T(s) = \frac{1}{1+G_{loop}(s)}$

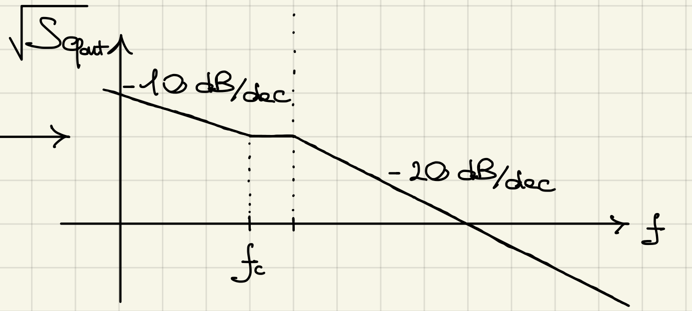




flicker noise corner frequency

typical VCO noise profile

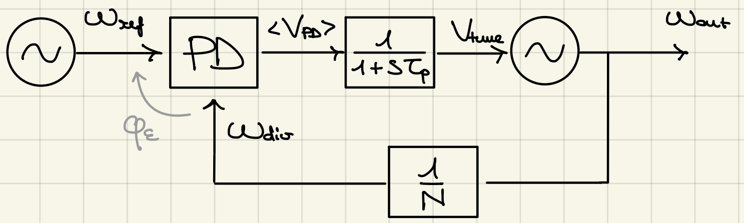
$$\xrightarrow{\frac{\varphi_{out}(s)}{\varphi_{vn}(s)}}$$



filtered output noise profile

$\Longrightarrow$ VCO noise is not well filtered!
(LF components are still relevant)

2) Static phase error



At steady-state:

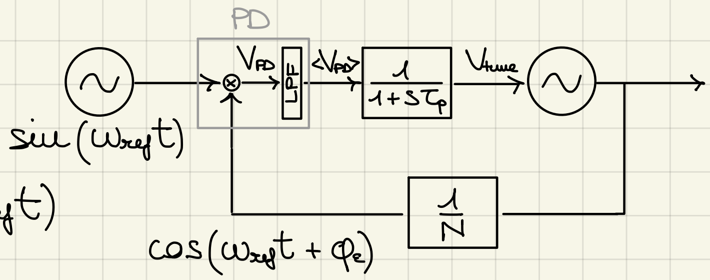
$$\omega_{out} = N\,\omega_{ref}$$
$$= \omega_{FR} + K_{VCO}\,V_{tune}$$

$$\langle V_{PD}\rangle = V_{tune} = K_{PD}\,\varphi_\varepsilon$$

$$\Longrightarrow \quad \varphi_\varepsilon = \frac{N\omega_{ref} - \omega_{FR}}{K_{VCO}\cdot K_{PD}} \neq 0$$

(φ_ε is parameter dependent i.e. it may vary with temperature, aging etc.)

3) Reference spurs

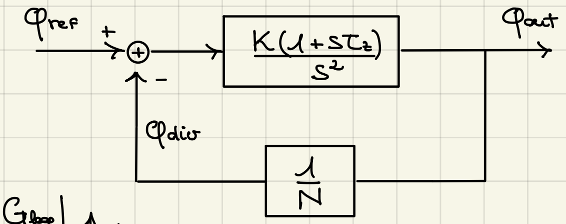


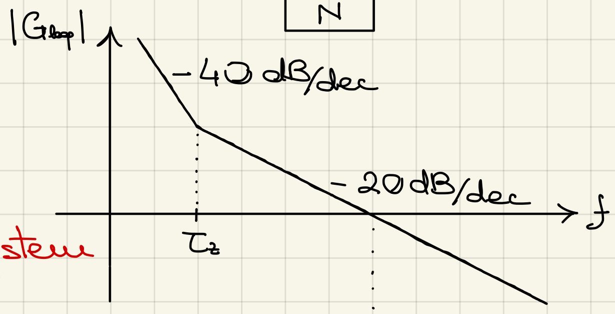
$$V_{PD} = \frac{1}{2}\sin(\varphi_\varepsilon) + \frac{1}{2}\sin(2\omega_{ref}t)$$

We want to remove the HF components from the PD output, since the LPF will attenuate them but won't completely cancel them.

⟹ Type-II PLLs solve these 3 issues

Equivalent model:





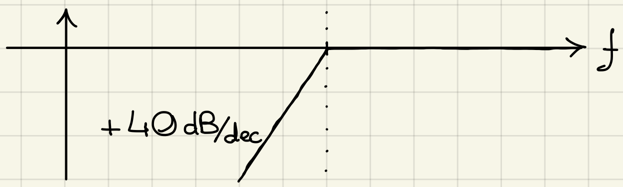
$$G_{loop}(s) = \frac{K(1+s\tau_z)}{\underbrace{s^2}_{\text{type-II system}}}\cdot\frac{1}{N}$$

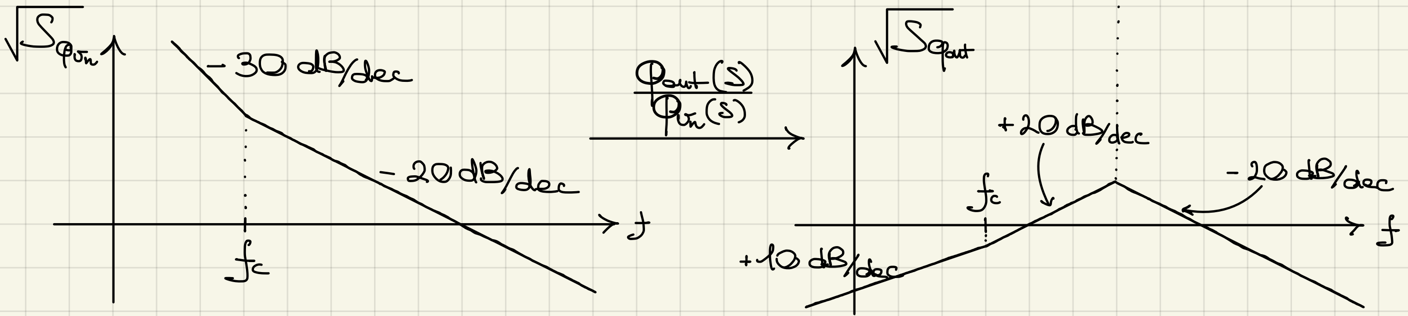
1) Better VCO noise filtering

$$\frac{\Phi_{out}(s)}{\Phi_{in}(s)} = 1 - T(s) =$$

$$= \frac{1}{1+G_{loop}(s)} = \quad K' = \frac{K}{N}$$

$$= \frac{1}{1+\frac{K'(1+s\tau_z)}{s^2}} = \frac{s^2}{s^2+sK'\tau_z+K'}$$





typical VCO noise profile filtered output noise profile

2) Zero static phase error

$$\frac{\Phi_\varepsilon(s)}{\Phi_{ref}(s)} = \frac{1}{1+G_{loop}(s)} = 1 - T(s) = \frac{s^2}{s^2+sK'\tau_z+K'}$$

Let's apply an input frequency step: $\Omega_{ref}(s) = \Delta\omega/s$

$\Phi_{ref}(s) = \Delta\omega/s^2$

$$\varphi_e(t^\infty) = \lim_{s\to 0} s \, \frac{\Delta\omega}{\cancel{s^2}} \cdot \frac{\cancel{s^2}}{s^2 + sK'\tau_z + K'} = 0$$ (as expected from previous discussion)

3) No reference spurs

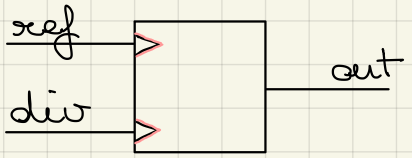
At steady-state $\varphi_e = 0 \implies$ we can build a phase detector such that:

$\text{out} = 0$ when $\varphi_e = 0$

instead of just:

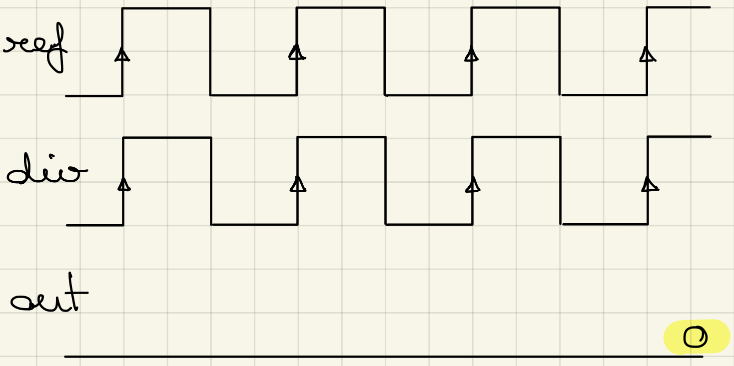
$\langle \text{out} \rangle = 0$ when $\varphi_e = 0$

Tri-state phase detector (PFD - Phase/Frequency Detector)

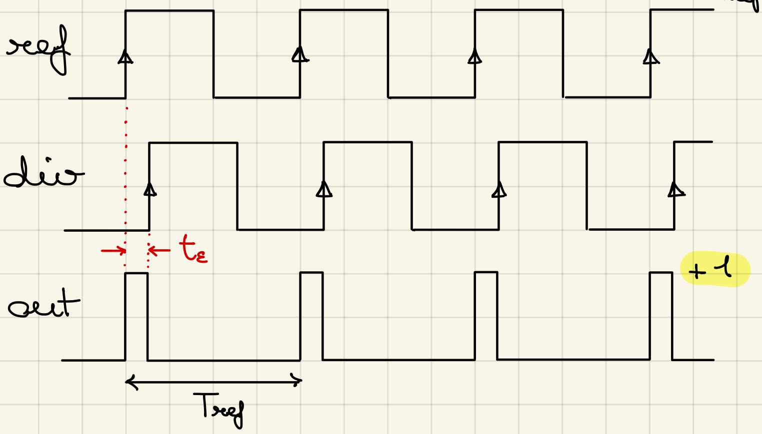


rising-edge sensitive

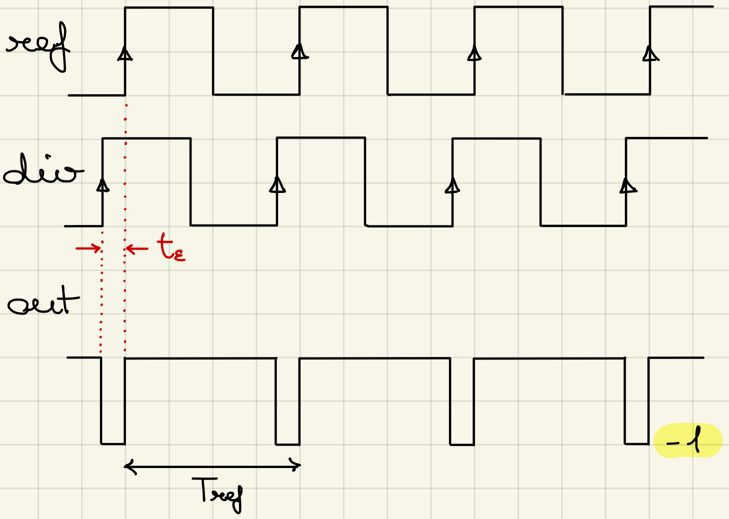
$t_\varepsilon = 0 \longrightarrow \text{out} \equiv 0$



$t_\varepsilon > 0$ (ref leads div) $\longrightarrow \langle \text{out} \rangle = \frac{t_\varepsilon}{T_{ref}}$

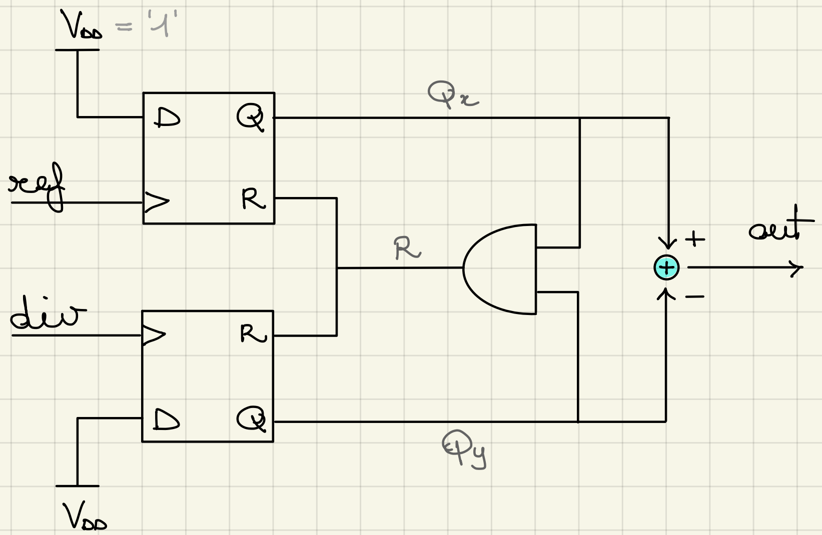


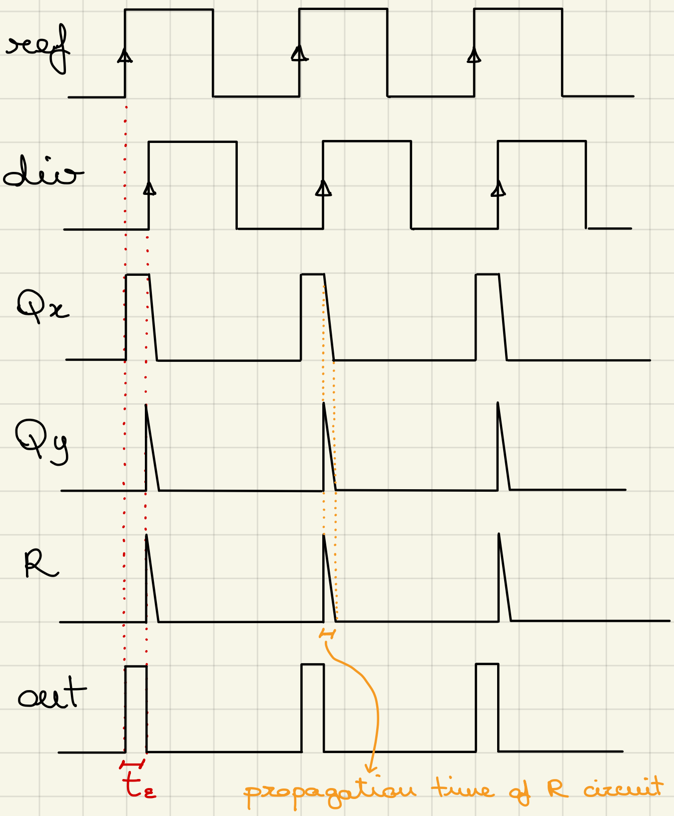
$t_\varepsilon > 0$ (ref follows div) $\longrightarrow \langle \text{out} \rangle = -\frac{t_\varepsilon}{T_{ref}}$



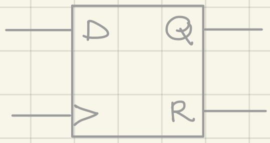
tri-state

Implementation of the PFD:



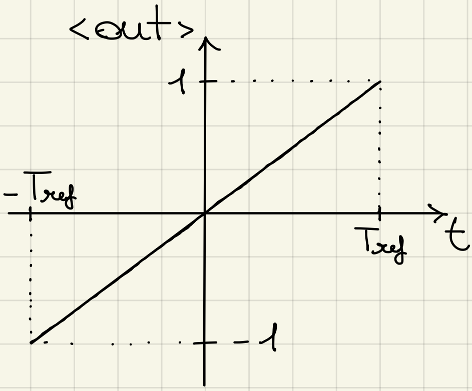


D flip-flop



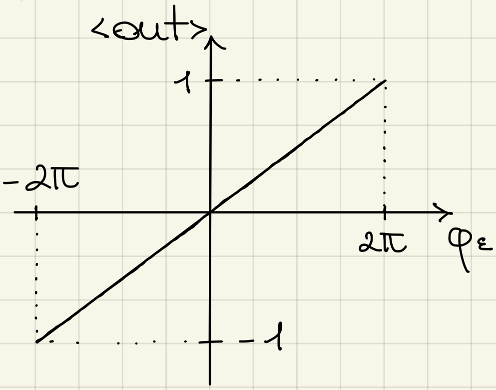
D	CK	R	Q
0	↑	X	0
1	↑	X	1
X	X	↑	0

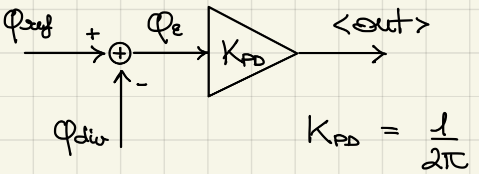
Same happens for $t_\varepsilon < 0$.



Static PFD characteristic:

$$\varphi_\varepsilon = \frac{2\pi}{T_{ref}} \cdot t_\varepsilon \qquad \langle out \rangle = \frac{t_\varepsilon}{T_{ref}} = \frac{\varphi_\varepsilon}{2\pi}$$

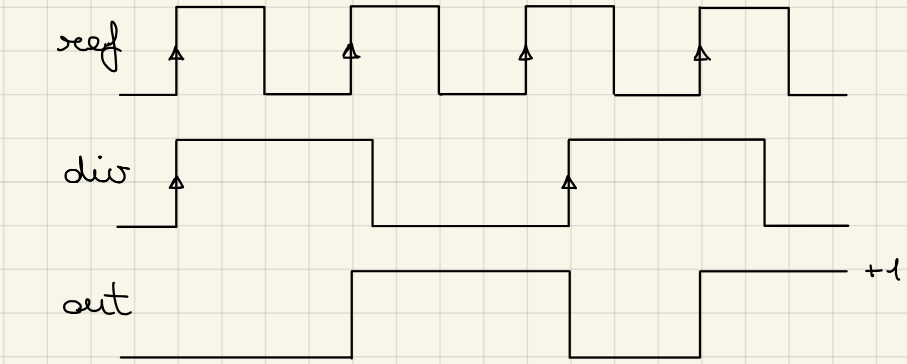




$$K_{PD} = \frac{1}{2\pi}$$

So why is it called phase/frequency detector?

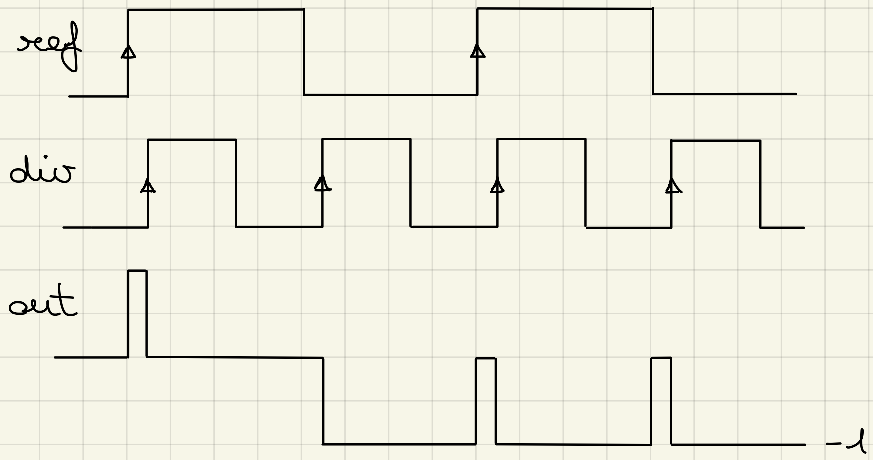
Consider for example during start-up when $\omega_{ref} \gg \omega_{div}$. Then the output will be at +1 for most of the time since rising edges of "ref" are much more frequent. So "out" is not proportional to t_ε however it provides a positive value so that the loop is forced to increase ω_{div}.



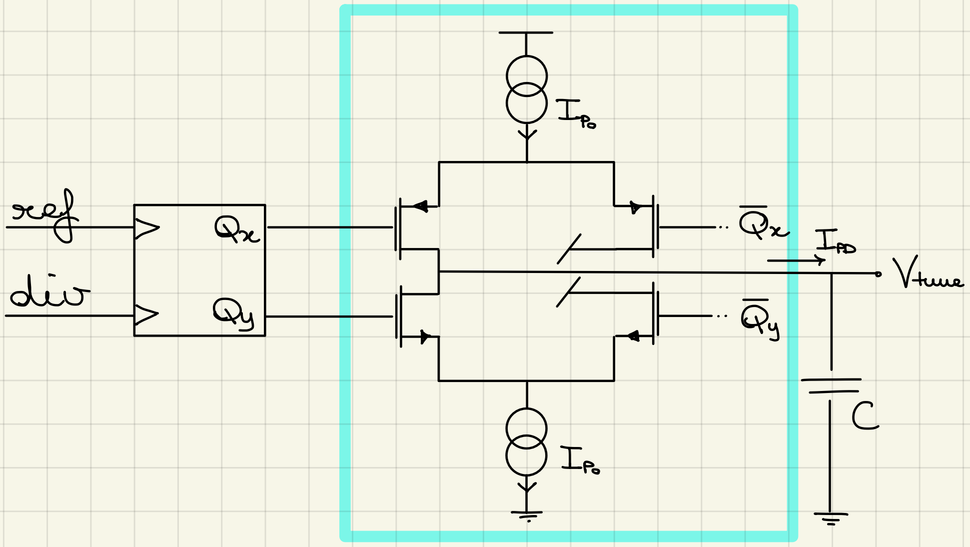
ω_{ref}
$\gg$
ω_{div}
$\downarrow$
$\langle out \rangle \approx 1$

In the opposite situation:

ω_{ref}
$\ll$
ω_{div}
$\downarrow$
$\langle out \rangle \approx -1$



Now how can we obtain the summing node in the PFD implementation? ⟶ Charge Pump



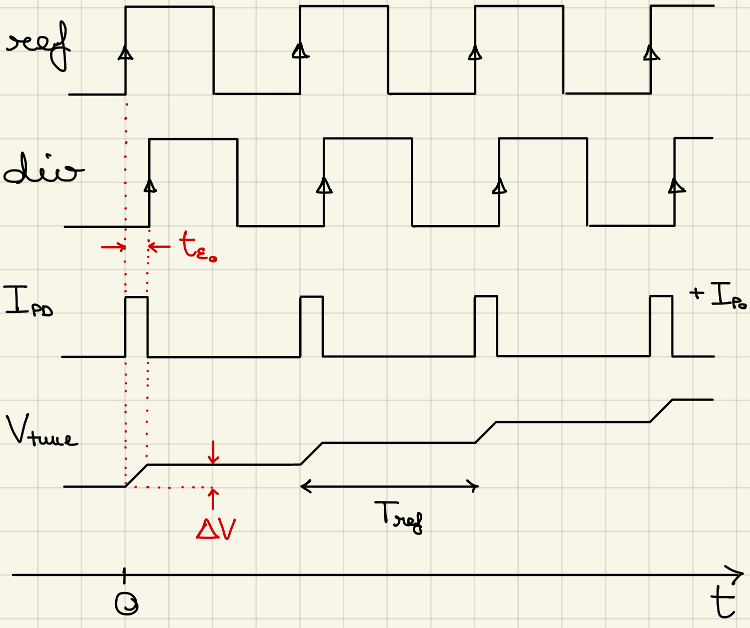
- Sum of PDF out with current
- High output impedence
  $\downarrow$
  connect C to perform integration without the need of OPAMP

- Equivalent model of the PFD

Consider to apply a t_{ε_0} step @ $t=0$

$$\varphi_{\varepsilon_0} = \frac{2\pi}{T_{ref}} t_{\varepsilon_0} \qquad \varphi_\varepsilon(s) = \frac{\varphi_{\varepsilon_0}}{s}$$

Evaluate $V_{tune}(s)$ to find $\frac{V_{tune}(s)}{\varphi(s)}$.



$$\frac{dV_{tune}}{dt} = \frac{I_{P0}}{C}$$

$$\Delta V = t_{\varepsilon}\frac{I_{P0}}{C}$$

$$\boxed{\text{Gardner's limit:}\quad BW_{PLL} < \frac{f_{ref}}{20}}$$

$$\downarrow$$

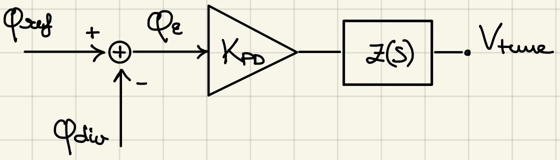
V_{tune} can be approximated as a ramp:

$$V_{tune}(t) \simeq \frac{\Delta V}{T_{ref}}\cdot t = \frac{t_{\varepsilon_0} I_{P0}}{C\,T_{ref}}\cdot t$$

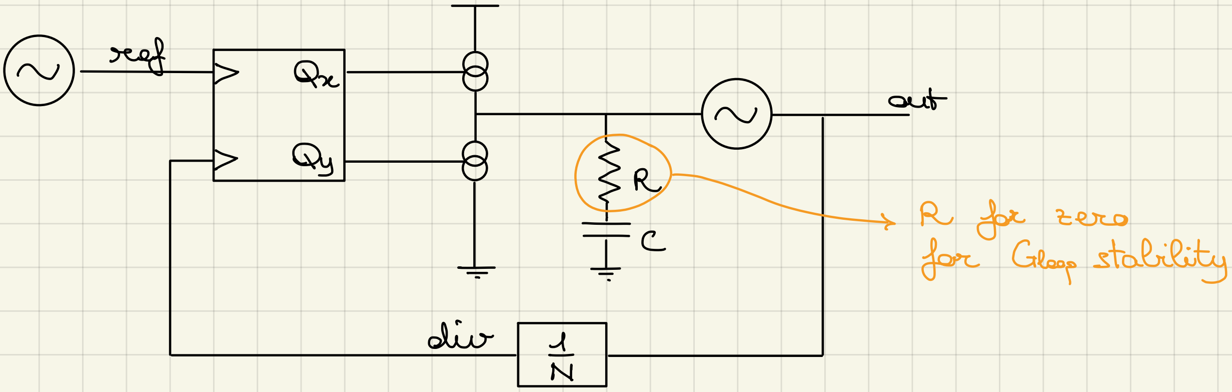
$$= \frac{I_{P0}}{C}\cdot\frac{\varphi_{\varepsilon_0}}{2\pi}\cdot t \qquad (t>0)$$

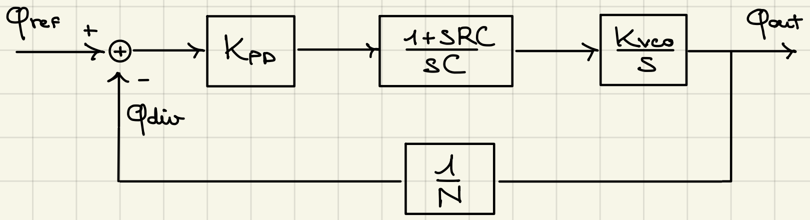
$$\longrightarrow \quad V_{tune}(s) = \frac{I_{P0}}{C}\,\frac{\varphi_{\varepsilon_0}}{2\pi}\,\frac{1}{s^2} = \underbrace{\left(\frac{I_{P0}}{2\pi}\right)}_{K_{PD}}\cdot\underbrace{\left(\frac{1}{sC}\right)}_{\substack{Z(s)\\ \text{connected}\\ \text{to the CP}}}\cdot\underbrace{\left(\frac{\varphi_{\varepsilon_0}}{s}\right)}_{\text{input step}}$$

$$\Longrightarrow \quad \boxed{\frac{V_{tune}(s)}{\varphi_{\varepsilon}(s)} = K_{PD}\cdot Z(s)} = \frac{I_{P0}}{2\pi}\cdot\frac{1}{sC}$$



Complete implementation of a type-II PLL with PFD:



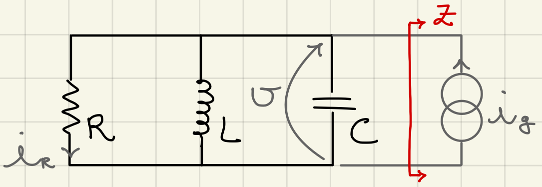


Passive Networks

"to obtain voltage/current amplification without active components"

Note: at RF freq. it is possible to implement integrated inductors

① Resonant circuits



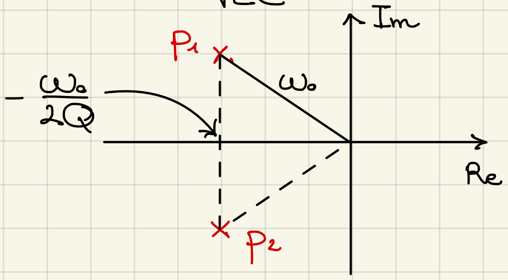
parallel RLC

Impedence: $Z = \frac{V}{I_g} = \frac{I_R \cdot R}{I_g} = H(s)R$

$$\frac{I_R}{I_g} = H(s) = \frac{1/R}{1/R + 1/sL + sC} = \frac{s\,\omega_0/Q}{\omega_0^2 + s\,\boxed{\omega_0/Q} + s^2}$$

$\omega_0/Q \rightarrow 2\xi\omega_0$

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad Q = \omega_0 RC = \frac{R}{\omega_0 L} = \sqrt{\frac{C}{L}} \cdot R$$



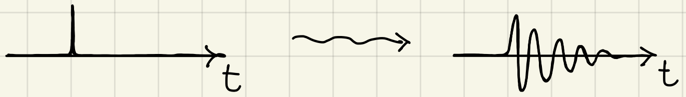
$$p_{1-2} = -\frac{\omega_0}{2Q} \pm j\omega_0\sqrt{1 - 1/4Q^2}$$

$$\xi = \frac{1}{2Q} = \frac{1}{2\omega_0 RC}$$

Meaning of Q factor:

1. inversely proportional to damping factor ξ

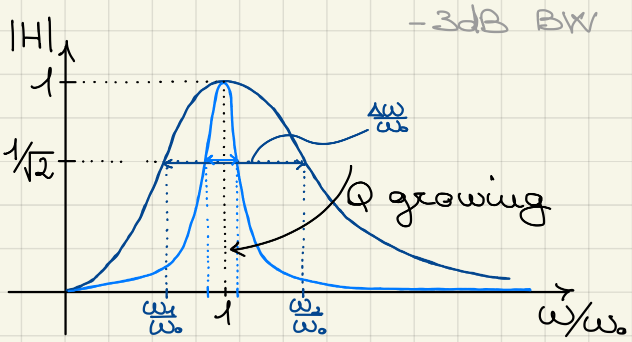
ξ small ⇔ Q large ⇔ underdamped poles



2. $H(j\omega) = \dfrac{j\,\omega\omega_0/Q}{\omega_0^2 + j\omega\omega_0/Q - \omega^2} = \dfrac{1}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$

Impose $|H(j\omega)|^2 = \frac{1}{2} \longrightarrow \dfrac{1}{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2} = \frac{1}{2}$

-3dB BW



$$Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2 = 1$$

$$Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = \pm 1$$

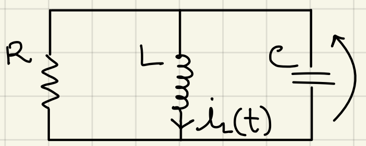
$$\omega^2 \mp Q\omega_0\omega - \omega_0^2 = 0$$

$$\omega_{1-2} = \omega_0\left(\mp\frac{1}{2Q} + \sqrt{1 + \frac{1}{4Q^2}}\right)$$

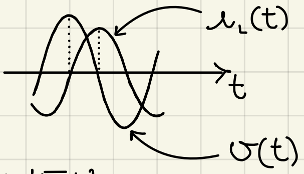
$$\boxed{\frac{\Delta\omega}{\omega_0} = \frac{\omega_2-\omega_1}{\omega_0} = \frac{\omega_0/2Q + \omega_0/2Q}{\omega_0} = \frac{1}{Q}}$$

Q is the ratio between the center frequency and the -3dB bandwidth of the frequency response.

3. energy meaning $\quad Q = \omega_0 RC = \omega_0 \frac{\mathcal{E}_{stored}}{P_{diss}}$



$$v(t) = \mathrm{Re}\{\overset{V\cdot e^{j\varphi_0}}{\overset{\parallel}{\bar{V}}}\, e^{j\omega_0 t}\}$$



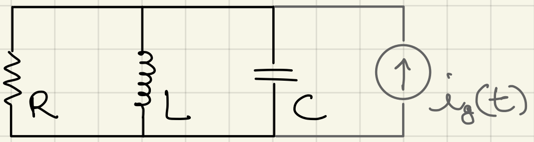
$$\mathcal{E}_{stored} = \frac{1}{2} L i_L^2(t) + \frac{1}{2} C v^2(t) = \frac{1}{2} C |\bar{V}|^2 \qquad P_{diss} = \frac{1}{2}\frac{|\bar{V}|^2}{R}$$

Q is ω_0 times the ratio between stored energy and dissipated power in a resonator

$$Q = \omega_0 \frac{\mathcal{E}_{stored}}{P_{diss}} = 2\pi \cancel{f_0} \frac{\mathcal{E}_{stored}}{\mathcal{E}_{\frac{diss}{cycle}} \cdot \cancel{f_0}} = 2\pi \frac{\mathcal{E}_{stored}}{\mathcal{E}_{diss/cycle}}$$

Q is also 2π times the ratio between stored and dissipated energy in each cycle

4 amplification at resonance



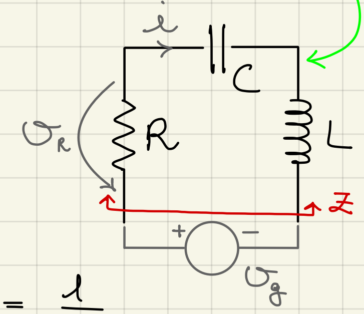
$i_g(t)$ sinusoidal at resonance $\omega_0 = \frac{1}{\sqrt{LC}}$

$j\omega_0 L \,/\!/\, \frac{1}{j\omega_0 C} \approx \infty$

$$|I_c| = \omega_0 C \cdot |V| = \omega_0 C\, |I_g| \cdot R = Q\, |I_g|$$

Q is the current gain between input current and capacitor/inductor current

Same arguments are valid for series RLC

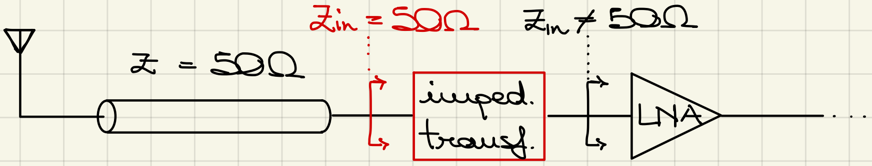


$$\text{Impedance:} \quad Z = \frac{V_g}{I} = R\frac{V_g}{V_R} = \frac{R}{K(s)}$$

$$\frac{V_R}{V_g} = K(s) = \frac{R}{R + sL + 1/sC} = \frac{s\,\omega_0/Q}{\omega_0^2 + s\,\omega_0/Q + s^2}$$

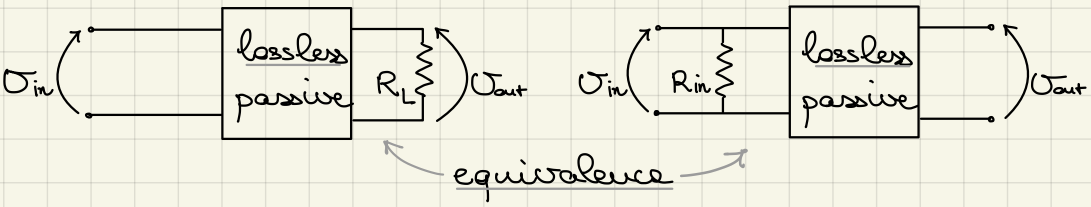
$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad Q = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R} = \sqrt{\frac{L}{C}}\cdot\frac{1}{R} = \frac{1}{2\xi}$$

Impedance transformation (* matching networks *)



Upward/Downward impedance transformation to avoid signal reflection (= power loss).

In general:



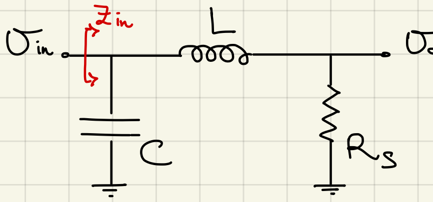
To have the equivalence hold:

$$\frac{1}{2}\frac{|V_{in}|^2}{R_{in}} = \frac{1}{2}\frac{|V_{out}|^2}{R_L} \Rightarrow \left[R_{in} = \frac{R_L}{\frac{|V_{out}|^2}{|V_{in}|^2}} = \frac{R_L}{G^2} \right]$$

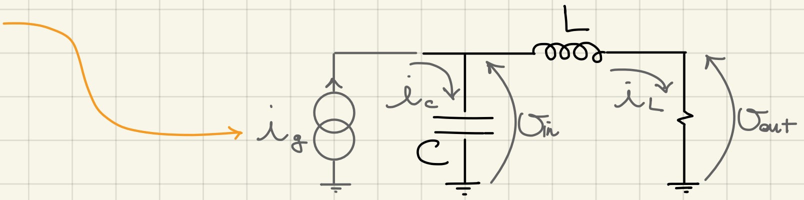
$G > 1$: amplification ⇒ DOWNWARD transf.

$G < 1$: attenuation ⇒ UPWARD transf.

* Upward L-match network (simplest network)



• Lossless approximation: $R_s \simeq 0$



At resonance: $|I_L| \simeq Q_L |I_g| \simeq |I_C|$ where $\boxed{Q_L = \frac{\omega_0 L}{R_s}} \gg 1$

$$|V_{out}| = |I_L| \cdot R_s = |I_C| R_s =$$

$$= \omega_0 C |V_{in}| \cdot R_s = \frac{|V_{in}|}{Q_L} \longrightarrow \text{attenuation}$$

fully real impedance

$R_{in} + j0$

$\parallel$

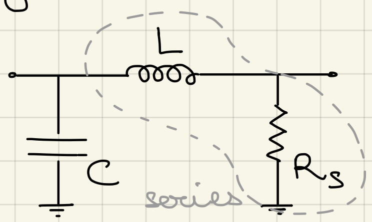
$$|Z_{in}| = \frac{|V_{in}|}{|I_g|} = \frac{|V_{out}| Q_L}{|I_L|/Q_L} = Q_L^2 R_s \longrightarrow \boxed{Z_{in} \cong Q_L^2 R_s}$$

$$\uparrow \; |I_L| = \frac{|V_{out}|}{R_s}$$

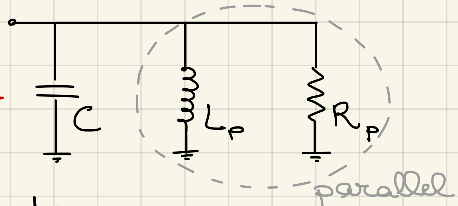
at $\omega_0 = \frac{1}{\sqrt{LC}}$, lossless approx.

$R_{in} \gg R_s \longrightarrow$ UPWARD impedance transformation

- General case (no lossless approx.)



equivalent around resonance



series-to-parallel transformation:

equivalence valid for any fixed ω!

frequency dependent!

$$j\omega L + R_s = \frac{j\omega L_p R_p}{j\omega L_p + R_p}$$

$$R_s(1+jQ_L) = \frac{j\omega L_p R_p}{j\omega L_p + R_p}$$ where $\boxed{Q_L = \frac{\omega L}{R_s}}$

$$R_s(1+jQ_L)(j\omega L_p + R_p) = j\omega L_p R_p$$

Re: $R_sR_p - R_sQ_L\omega L_p = 0$

Im: $R_sQ_LR_p + R_s\omega L_p = \omega L_p R_p$

$$\omega_0 L_p = \frac{R_p}{Q_L}$$

$$R_sQ_L\cancel{R_p} + R_s\frac{\cancel{R_p}}{Q_L} = \frac{R_p}{Q_L}\cancel{R_p}$$

$$\boxed{\begin{aligned} R_p &= R_s(1+Q_L^2) \\ L_p &= L\,\frac{1+Q_L^2}{Q_L^2} \end{aligned}}$$

it has a slight shift from the lossless approx. due to both R_p and L_p

$$\boxed{Z_{in} = R_p = (1+Q_L^2)R_s}$$

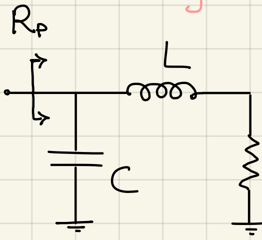
at $\omega_0 = \frac{1}{\sqrt{L_pC}}$, no approx.

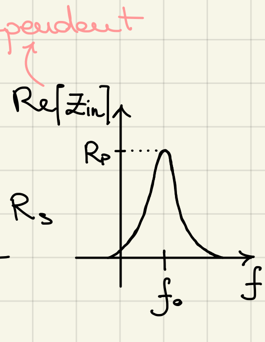
not L!!!

L-match network design rules

ω_0, R_s and R_p known

transformation ratio R_p/R_s



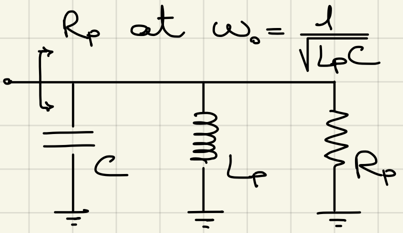


1. $R_p = R_s(1+Q_L^2) \Longrightarrow Q_L = \sqrt{\frac{R_p}{R_L} - 1}$

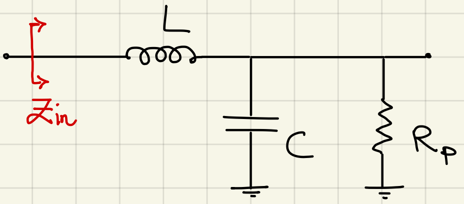
large transformation ⟹ narrowband transformation

2. $Q_L = \frac{\omega_0 L}{R_s} \Longrightarrow L$

3. $\omega_0 = \frac{1}{\sqrt{L_pC}}$ and $L_p = L\,\frac{1+Q_L^2}{Q_L^2} \Longrightarrow C$



* Downward L-match network

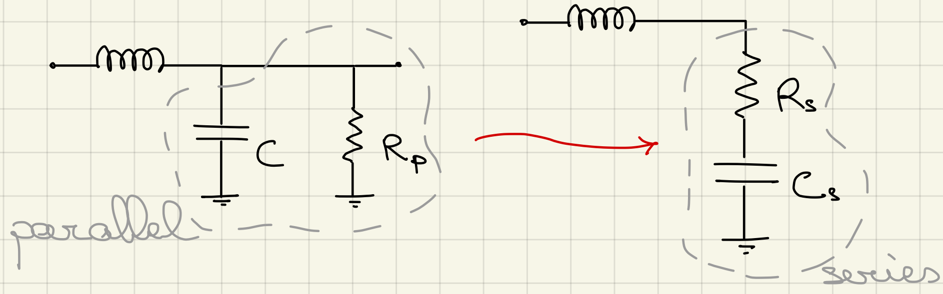


• Lossless approximation: $R_p \simeq \infty$

$|V_{out}| = Q_c |V_{in}|$ where $\boxed{Q_c = \omega_0 C R_p} \gg 1$

$$\longrightarrow \boxed{Z_{in}(j\omega_0) \simeq \frac{R_p}{Q_c^2}} \qquad \omega_0 = \frac{1}{\sqrt{CL}}$$

• General case: parallel-to-series transformation (around resonance)

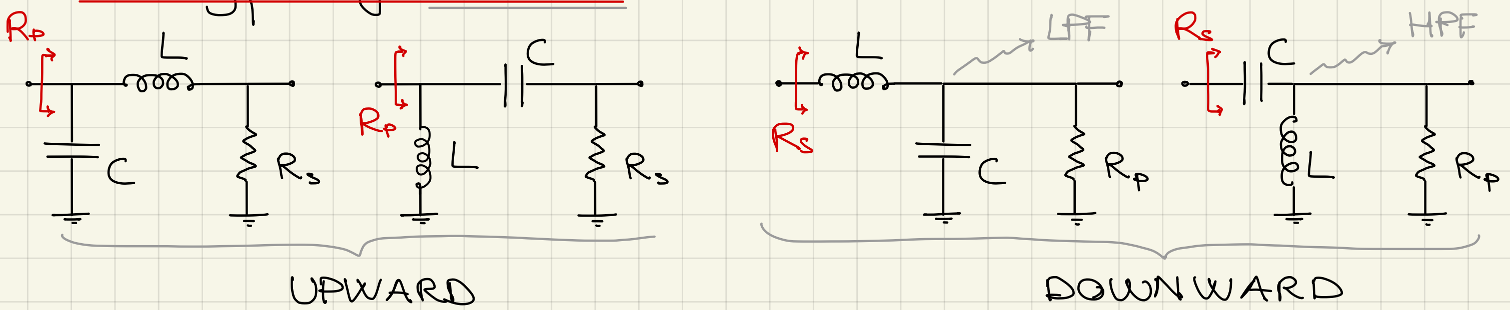


$$\boxed{\begin{aligned} R_s &= \frac{R_p}{1+Q_c^2} \\ C_s &= C\,\frac{1+Q_c^2}{Q_c^2} \end{aligned}}$$

$$\Downarrow$$

$$\boxed{Z_{in}(j\omega_0) = R_s} \qquad \omega_0 = \frac{1}{\sqrt{C_s L}}$$

All types of L-match



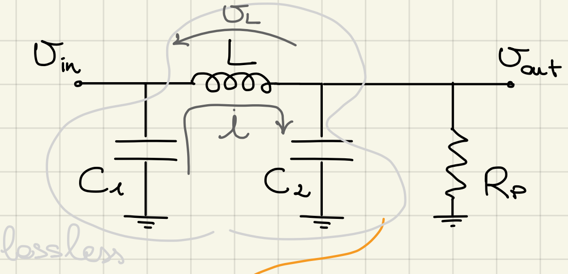
Choice criteria:
- frequency response
- DC blocking
- absorption of stray capacitances

Basic relations for any type of transformation:

$$\boxed{R_p = R_s\left(1+Q^2\right) \qquad X_p = X_s\left(1+\frac{1}{Q^2}\right) \qquad Q = \frac{X_s}{R_s} = \frac{R_p}{X_p}}$$

where $X = \frac{1}{\omega_0 C} = \omega_0 L$

* π-match network (or "Colpitts" network)

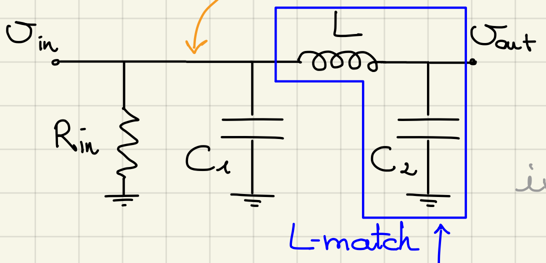


- Lossless approximation: $R_p \simeq \infty$

$$I = sC_2V_{out} \cong -sC_1V_{in}$$

$$\downarrow$$

$$\frac{V_{out}}{V_{in}} \cong -\frac{C_1}{C_2}$$



$$\underbrace{\frac{1}{2}\frac{|V_{in}|^2}{R_{in}}}_{\text{power diss in equ. circ.}} = \underbrace{\frac{1}{2}\frac{|V_{out}|^2}{R_p}}_{\text{power diss. in orig. circ.}} \rightarrow R_{in} = R_p\left(\frac{V_{in}}{V_{out}}\right)^2$$

$$\boxed{R_{in} \cong R_p\left(\frac{C_2}{C_1}\right)^2}$$

$C_2 > C_1$ UPWARD

$C_2 < C_1$ DOWNWARD

$Q = ?$

$$Q = \omega_0\frac{\mathcal{E}_{stored}}{P_{diss}} \cong \boxed{\omega_0 R_p C_2}\boxed{\left(1+\frac{C_2}{C_1}\right)}$$

↳ enhancement factor

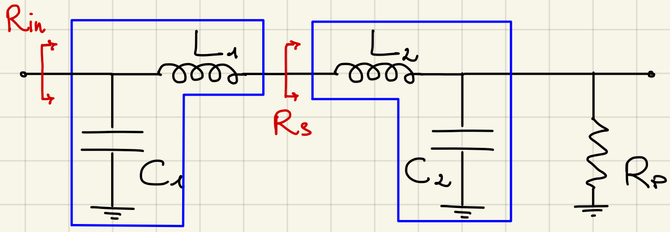
$$\begin{cases} \mathcal{E}_{stored} = \frac{1}{2}\frac{C_1C_2}{C_1+C_2}|V_L|^2 \\ P_{diss} = \frac{1}{2}\frac{|V_{out}|^2}{R_p} \overset{\uparrow}{\cong} \frac{1}{2}\left(\frac{C_1}{C_1+C_2}\right)^2\frac{|V_L|^2}{R_p} \end{cases}$$

$$V_L = V_{in} - V_{out} \cong -\frac{C_2}{C_1}V_{out} - V_{out} = -V_{out}\left(\frac{C_1+C_2}{C_1}\right)$$

⇒ Q factor of π-network > Q factor of L-network

- General case

$$L_1 + L_2 = L$$



$$R_s = \frac{R_p}{1+Q_2^2} \quad \text{where } Q_2 = \omega_0 R_p C_2 = \frac{\omega_0 L_2}{R_s}$$

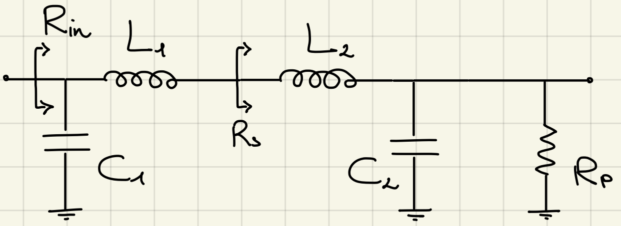
$$R_{in} = R_s\left(1+Q_1^2\right) \quad \text{where } Q_1 = \frac{\omega_0 L_1}{R_s}$$

UPWARD L-match + DOWNWARD L-match

$$\boxed{R_{in} = R_p\frac{1+Q_1^2}{1+Q_2^2}} \qquad \frac{R_{in}}{R_p} = \frac{1+Q_1^2}{1+Q_2^2}$$

π-match network design rules

ω_0, R_p, R_{in} and Q known



1. $Q = \frac{\omega_0 (L_1 + L_2)}{R_s} = Q_1 + Q_2 = \sqrt{\frac{R_{in}}{R_s} - 1} + \sqrt{\frac{R_P}{R_s} - 1} \Longrightarrow R_s$

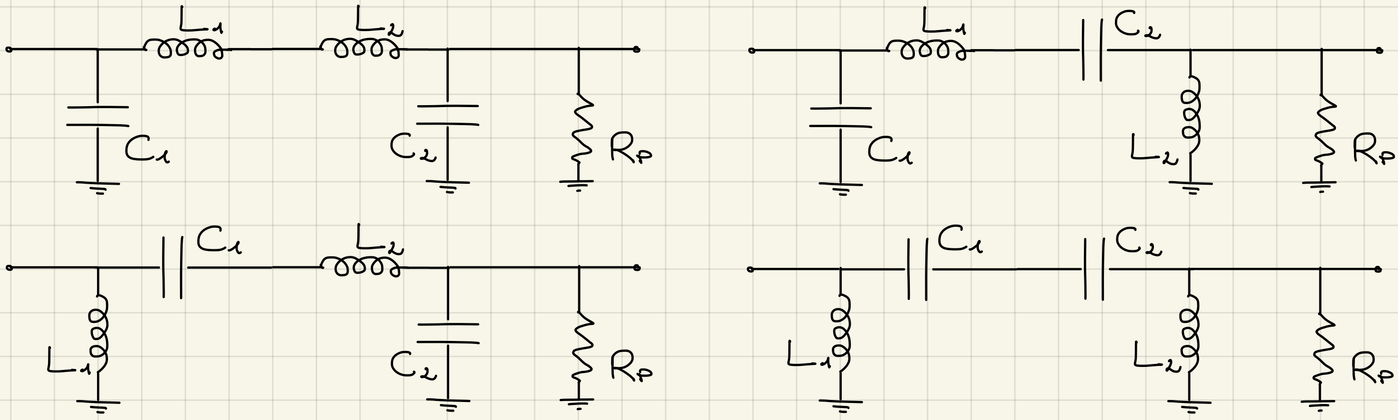
2. $L_1 + L_2 = \frac{Q \cdot R_s}{\omega_0} \Longrightarrow L$

3. $Q_2 = \omega_0 R_P C_2 \Longrightarrow C_2$

4. $Q_1 = \frac{\omega_0 L_1}{R_s} \Longrightarrow L_1 \Longrightarrow L_2$

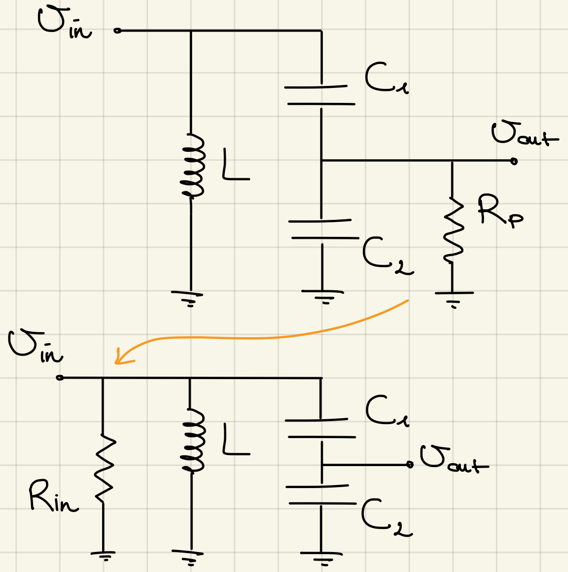
5. $\omega_0 = \frac{1}{\sqrt{L_2 C_2 \frac{1+Q_2^2}{Q_2^2}}} = \frac{1}{\sqrt{L_1 C_1 \frac{1+Q_1^2}{Q_1^2}}} \Longrightarrow C_1$

All types π-match networks



(always UPWARD + DOWNWARD)

* Resonator with tapped capacitor (or inductor)



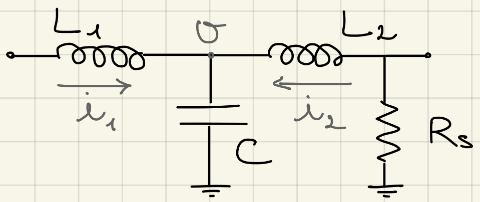
- Lossless approximation: $R_P \simeq \infty$

$$\frac{V_{out}}{V_{in}} \simeq \frac{C_1}{C_1 + C_2} \qquad \frac{1}{2} \frac{|V_{out}|^2}{R_p} = \frac{1}{2} \frac{|V_{in}|^2}{R_{in}}$$

$$\Longrightarrow \boxed{R_{in} \cong R_P \left(1 + \frac{C_2}{C_1}\right)^2}$$

UPWARD transformation

* T-match network



(and all other permutations)

- Lossless approximation: $R_s \simeq 0$

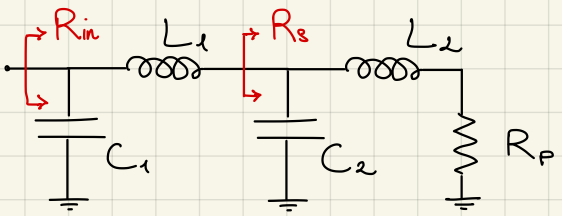
$-V = sL_1 I_1 \cong sL_2 I_2 \qquad \frac{1}{2} R_{in}(I_1)^2 = \frac{1}{2} R_s (I_2)^2$

$$\Longrightarrow \boxed{R_{in} \cong R_s \left(\frac{L_1}{L_2}\right)^2}$$

$L_1 > L_2$ UPWARD

$L_1 < L_2$ DOWNWARD

* Cascaded L-match network



UPWARD

(and all other permutations)

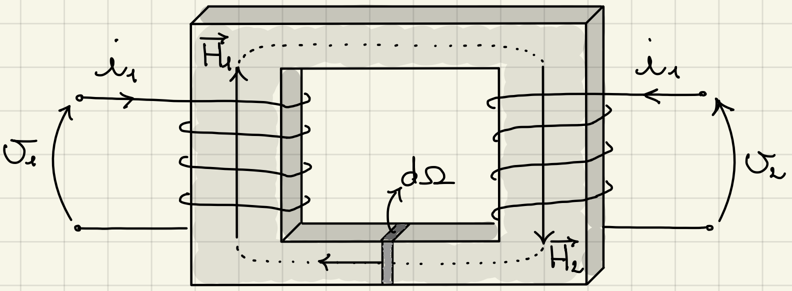
$R_{in} = R_s(1+Q_1^2)$ where $Q_1 = \frac{\omega_0 L_1}{R_s}$

$R_s = R_P(1+Q_2^2)$ where $Q_2 = \frac{\omega_0 L_2}{R_P}$

$$\boxed{R_{in} = R_P(1+Q_2^2)(1+Q_1^2)}$$

allows for larger bandwidth when performing large transform. compared to single L-match

② Inductor coupling (transformers)



$$\mathcal{E}_m = \frac{\mu}{2}\,|\overbrace{\vec{H}_1 + \vec{H}_2}^{\text{total H field}}|^2 \, d\Omega =$$

$$= \underbrace{\frac{\mu}{2}|\vec{H}_1|^2 d\Omega}_{\text{energy coil 1}} + \underbrace{\frac{\mu}{2}|\vec{H}_2|^2 d\Omega}_{\text{energy coil 2}} + \underbrace{\mu\, \vec{H}_1 \cdot \vec{H}_2\, d\Omega}_{\text{mutual energy} > 0}$$

$\vec{H}_1, \vec{H}_2$ same orientation*

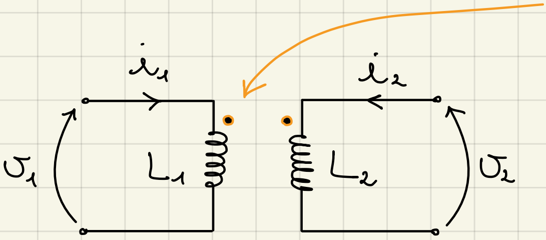
$\Rightarrow$ case of POSITIVE MUTUAL ENERGY

* depends on both 1) wire windings and 2) current direction

the dots indicate if the coupling is positive $(M>0)$ or negative $(M<0)$

both currents enter/exit the dots

only one current enters/exits the dot



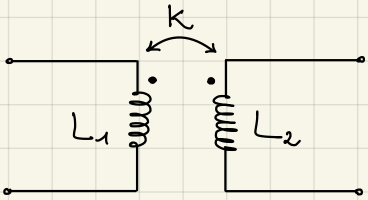
e.g. in this case it is POSITIVE

magnetic flux

$$\begin{cases}\phi_1 = L_1 i_1 + M i_2 \\ \phi_2 = M i_1 + L_2 i_2\end{cases} \qquad \begin{cases} v_1 = \dot{\phi}_1 \\ v_2 = \dot{\phi}_2\end{cases}$$

M is coupled inductance

$$\mathcal{E}_m = \int_0^t \underbrace{(v_1 i_1 + v_2 i_1)}_{\text{power}} dt' = \underbrace{\frac{1}{2} L_1 i_1^2}_{\text{energy coil 1}} + \underbrace{\frac{1}{2} L_2 i_2^2}_{\text{energy coil 2}} \overbrace{+ M}^{\text{POSITIVE mutual energy}} i_1 i_2$$



Coupling coefficient $K := \frac{|M|}{\sqrt{L_1 L_2}}$

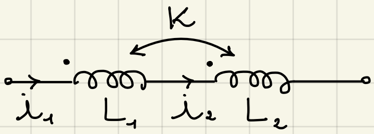
Conservation of energy implies that: $0 \leq K \leq 1$

no coupling

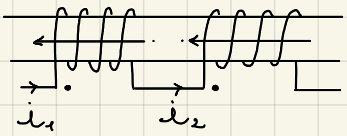
maximum coupling

Both ideal cases

Example: series of coupled inductors $i = i_1 = i_2$

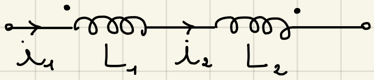


- POSITIVE M (mutual energy) because I_1 and I_2 both enter the dotted terminals
- Total inductance L_{tot}:



$$\phi = \phi_1 + \phi_2 = L_1 i_1 + M i_2 + L_2 i_2 + M i_1 = (L_1 + L_2 + 2M) i$$

$$\Longrightarrow L_{tot} = L_1 + L_2 + 2M \underset{\text{if } L_1 = L_2 = L}{=} 2L + 2K\sqrt{L^2} = 2L(1+K) \begin{cases} \xrightarrow{K=0} 2L \\ \xrightarrow{K=1} 4L \end{cases}$$



- NEGATIVE M

- $\Phi = \Phi_1 + \Phi_2 = L_1 i_1 - |M| i_2 + L_2 i_2 - |M| i_1 =$

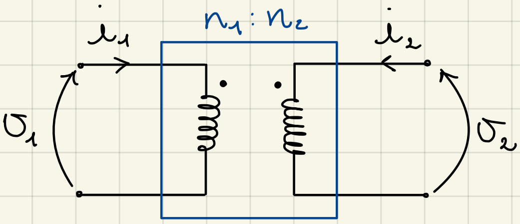
$$= (L_1 + L_2 - 2|M|)\, i$$

$$\Longrightarrow L_{tot} = L_1 + L_2 - 2|M| \underset{\uparrow}{=} 2L(1-K) \begin{cases} \xrightarrow{K=0} 2L \\ \xrightarrow{K=1} 0 \end{cases}$$

if $L_1 = L_2 = L$

Equivalent models of coupled inductors

- Model based on ideal transformer



ideal transformer

Hp: 1) No flux dispersion
(k = 1)

$\Phi_1 = n_1 \Phi$ ← flux of a single turn

$\Phi_2 = n_2 \Phi$

$\Longrightarrow \boxed{\frac{v_1}{v_2} = \frac{n_1}{n_2}}$ $n_2 > n_1$ → voltage amplification

2) Infinite self-inductance
($L_1, L_2 \longrightarrow \infty$)

reluctance $[H^{-1}]$

Hopkin's law: $m.m.f. = \Phi \cdot R = \frac{\Phi}{\Lambda}$

Ohm's law: $V = I \cdot R$

permeance [H]

Ampere's law: $m.m.f. = n_1 i_1 + n_2 i_2$

magnetomotive force

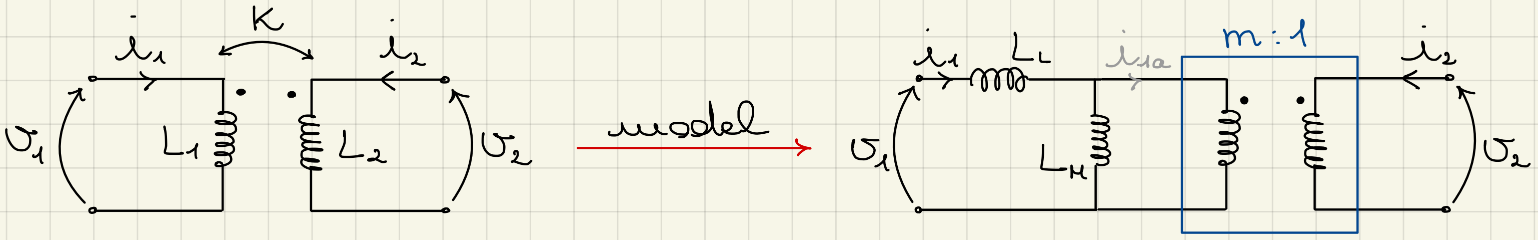
Because of infinite L: $R \longrightarrow 0$ $(\Lambda \longrightarrow \infty)$

$\Longrightarrow m.m.f. \longrightarrow 0$ $\Longrightarrow \boxed{\frac{i_1}{i_2} = -\frac{n_2}{n_1}}$ $n_1 > n_2$ → current amplification

$\Longrightarrow \frac{v_1}{v_2} \cdot \frac{i_1}{i_2} = \frac{n_1}{n_2}\left(-\frac{n_2}{n_1}\right) = -1 \longrightarrow \underline{v_1 i_1 + v_2 i_2 = 0}$

↓

ideal transformer is lossless



$L_L = (1-K^2)\,L_1$ "leakage" inductance

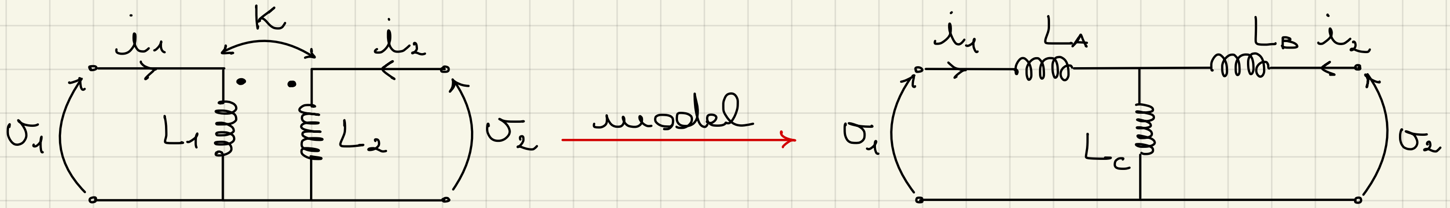
$L_M = K^2 \cdot L_1$ "magnetizing" inductance

$m = K\sqrt{\frac{L_1}{L_2}}$

Verification: $\Phi_1 = L_1 \ {}_1 + M \ {}_2 \rightarrow L_1 = \left.\frac{\Phi_1}{i_1}\right|_{i_2=0} \rightarrow L_1 = L_L + L_M$

$i_2 = 0 \rightarrow i_{1a} = 0 \rightarrow \Phi_1 = (L_L + L_M)\, i_1$

- T- circuit model



one end
must be joint
to use this model

$L_A = L_1 - M$

$L_B = L_2 - M$

$L_C = M$

Verification: $L_1 = \left.\frac{\Phi_1}{i_1}\right|_{i_2=0} = L_A + L_C \qquad L_2 = \left.\frac{\Phi_2}{i_2}\right|_{i_1=0} = L_B + L_C$

‖Oscillators‖

E.g.: VCO or CCO → electrically-tuned oscillators

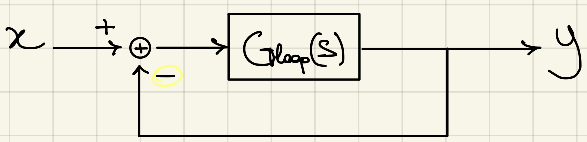
XO ⟶ crystal oscillator

Mathematical models: 1) feedback system

2) negative resistance

neg vs pos. is just a matter of definitions

1) a. Negative feedback



$$\frac{Y(s)}{X(s)} = \frac{G_{loop}(s)}{1+G_{loop}(s)}$$

Oscillation condition: $\{Y(j\omega_0) \neq 0 \text{ with } X(j\omega_0) = 0\}$

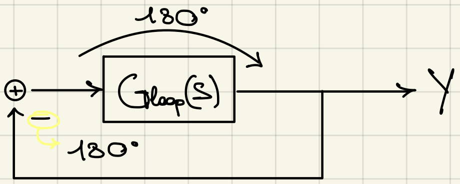
but then $\frac{Y(j\omega_0)}{X(j\omega_0)} = \frac{G_{loop}(j\omega_0)}{1+G_{loop}(j\omega_0)} \longrightarrow \infty \Longrightarrow \boxed{G_{loop}(j\omega_0) = -1}$

$s = j\omega_0$ is a solution of $G_{loop}(s) = -1$

$j\omega_0$ is a pole of the closed-loop system

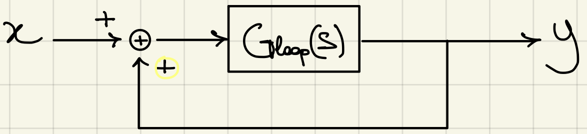
$$G_{loop}(j\omega_0) = -1 \iff \begin{cases} |G_{loop}(j\omega_0)| = 1 \\ \angle G_{loop}(j\omega_0) = \pm 180^\circ \end{cases}$$ Barkhausen's conditions

"autonomous system"



sinusoid at the output gets back through the loop with same amplitude and phase

b. Positive feedback



$$\frac{Y}{X} = \frac{G_{loop}(s)}{1-G_{loop}(s)} \longrightarrow \infty$$

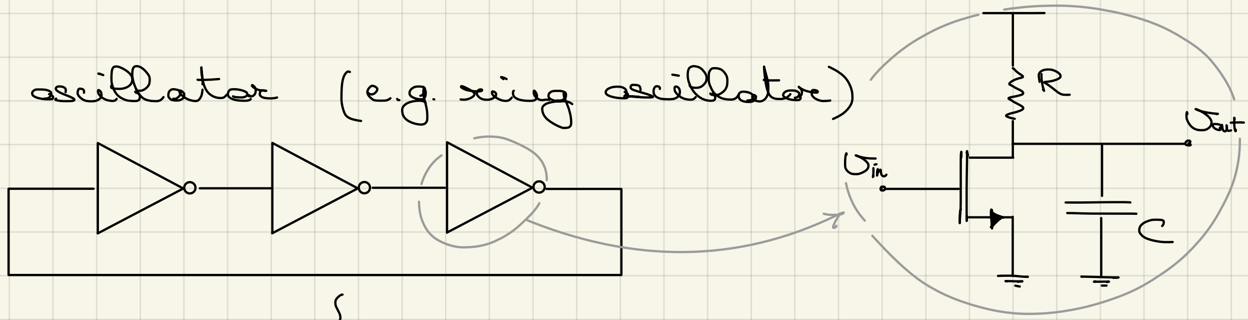
Oscillation condition:

$$\boxed{G_{loop}(j\omega_0) = +1}$$

$$\begin{cases} |G_{loop}(j\omega_0)| = 1 \\ \angle G_{loop}(j\omega_0) = 0^\circ / 360^\circ \end{cases}$$

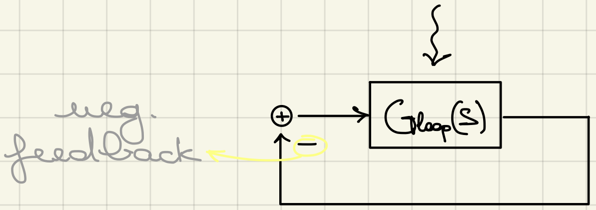
Examples:

- RC oscillator (e.g. ring oscillator)



$$\frac{V_{out}}{V_{in}} = -\frac{G}{1+s\tau}$$

$G > 0$
(simple linear model)



$$\boxed{G_{loop}(s) = \frac{G^3}{(1+s\tau)^3}}$$

Oscillation conditions:

1. $\measuredangle G_{loop}(j\omega_0) = -\pi$ ← could be $+\pi$ as well

$$\measuredangle \frac{G^3}{(1+j\omega_0\tau)^3} = -\pi$$

$$\underbrace{\cancel{\measuredangle G^3}}_{0} - 3\,\mathrm{arctg}(\omega_0\tau) = -\pi$$

$$\mathrm{arctg}(\omega_0\tau) = +\frac{\pi}{3}$$

$$\omega_0\tau = \sqrt{3}$$

$$\omega_0 = \frac{\sqrt{3}}{\tau}$$

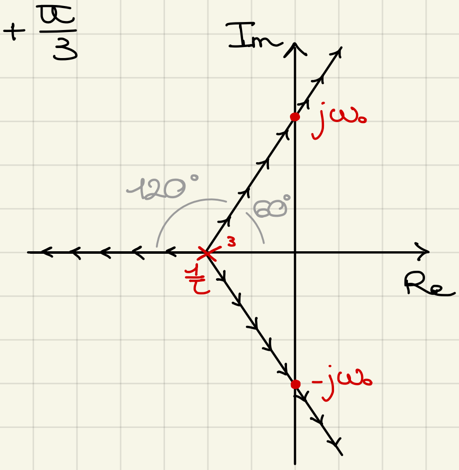
2. $|G_{loop}(j\omega_0)| = 1$

$$\frac{G^3}{[1+\underbrace{(\omega_0\tau)^2}_{\sqrt{3}}]^{3/2}} = 1$$

$$G^3 = (1+3)^{3/2}$$

$$G^3 = 2^3$$

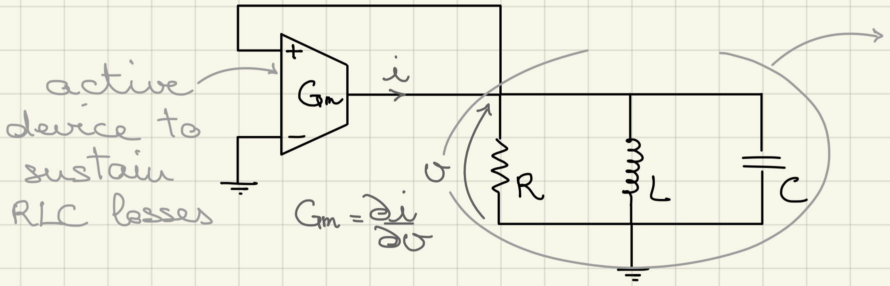
$$G = 2$$



Root locus

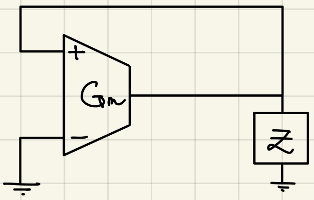
$$s = \pm j\omega_0 = \pm j\frac{\sqrt{3}}{\tau}$$

LC oscillator

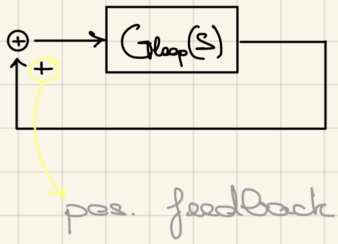


$$Z(s) = R\,\frac{s\,\omega_r/Q}{s^2 + s\frac{\omega_r}{Q} + \omega_0^2}$$

$$\omega_r = \frac{1}{\sqrt{LC}} \qquad Q = \omega_r RC$$



$\rightsquigarrow$



$$\boxed{G_{loop}(s) = G_m Z(s) = G_m R \frac{s\,\omega_r/Q}{s^2 + s\,\omega_r/Q + \omega_r^2}}$$

1. $\measuredangle G_{loop}(j\omega_0) = 0$

$$\measuredangle\left[\frac{j\omega_0\omega_r/Q}{(j\omega_0)^2 + j\omega_0\,\omega_r/Q + \omega_r^2}\right] = 0$$

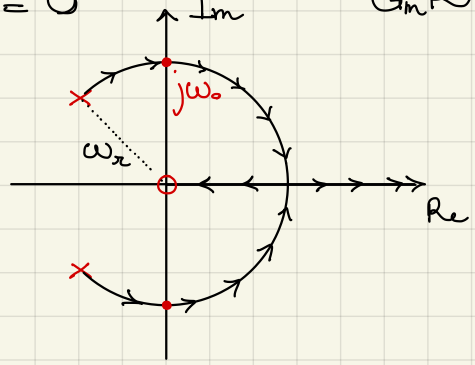
$$\frac{\pi}{2} - \text{arctg}\left(\frac{\omega_0\omega_r/Q}{\omega_r^2 - \omega_0^2}\right) = 0$$

$\frac{\pi}{2}$ when: $\omega_0 = \omega_r$

2. $|G_{loop}(s)| = 1$

$$G_m R \frac{\cancel{\omega_0\omega_r/Q}}{\sqrt{\underset{\to 0}{(\omega_r-\omega_0)^2} + \cancel{\left(\frac{\omega_0\omega_r}{Q}\right)^2}}} = 1$$

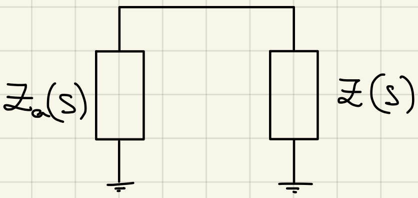
$$G_m R = 1$$



Root locus

$$s = \pm j\omega_0 = \pm j\omega_r$$

2)

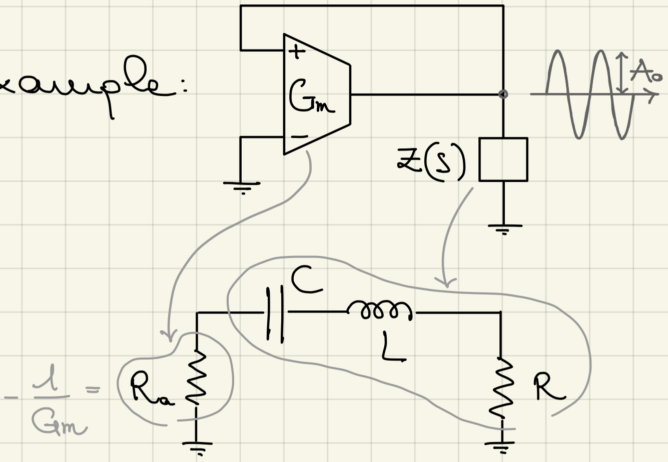


Oscillation condition:

balance between dissipated power and active power

$$\{Z_a(j\omega_0) + Z(j\omega_0) = 0\}$$

For example:



$$R_a + \frac{1}{j\omega_0 C} + j\omega_0 L + R = 0$$

$$\frac{1}{2}\frac{A_0^2}{R} = \frac{1}{2} G_m A_0^2$$

dissipated power — active power

$\downarrow$

$$G_m = \frac{1}{R}$$

same result obtained with feedback model, but evaluation is quicker

$$\longrightarrow R_a + R = 0, \quad \frac{1}{j\omega_0} + j\omega_0 L = 0 \longrightarrow R_a = -R, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

In general: $Z_a(j\omega_0) = -Z(j\omega_0) \longrightarrow \begin{cases} \mathrm{Re}[Z_a(j\omega_0)] = -\mathrm{Re}[Z(j\omega_0)] \\ \mathrm{Im}[Z_a(j\omega_0)] = -\mathrm{Im}[Z(j\omega_0)] \end{cases}$

To obtain a practical oscillator, we need an amplitude stabilization mechanism.

e.g.: LC oscillator

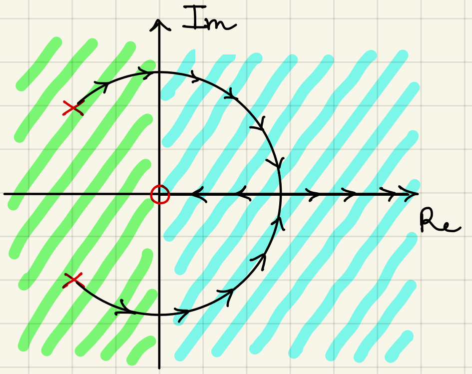
- $G_m R < 1 \longrightarrow$ poles in LHP

$$G_m \frac{A_0^2}{2} \lessgtr \frac{A_0^2}{2R}$$

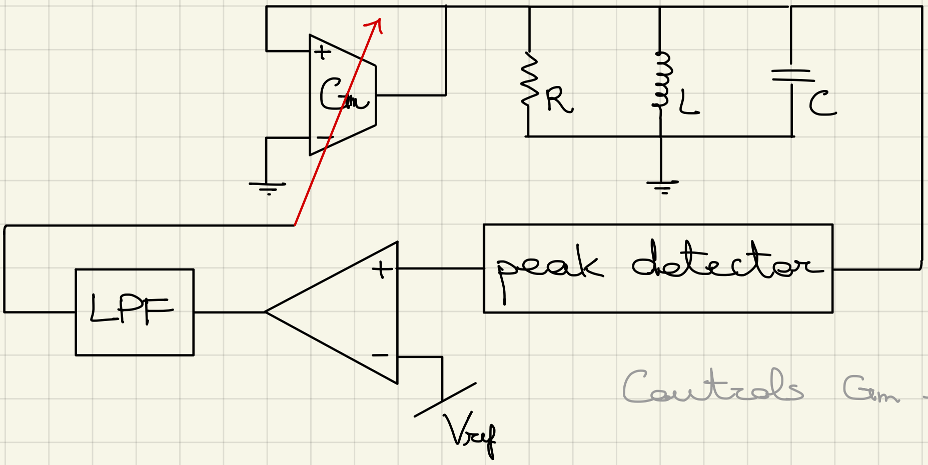
energy provided $\lessgtr$ dissipated energy

$$A_0 \nearrow 0 \qquad A_0 \searrow \infty$$

- $G_m R > 1 \longrightarrow$ poles in RHP



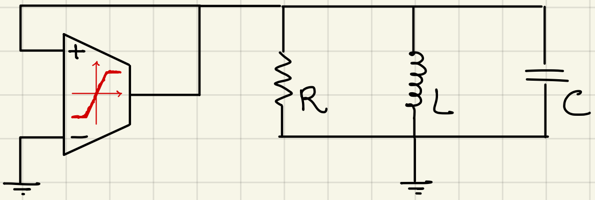
1) Automatic amplitude control (negative feedback)



Controls G_m i.e. Z_a

2) Non-linearity of active devices

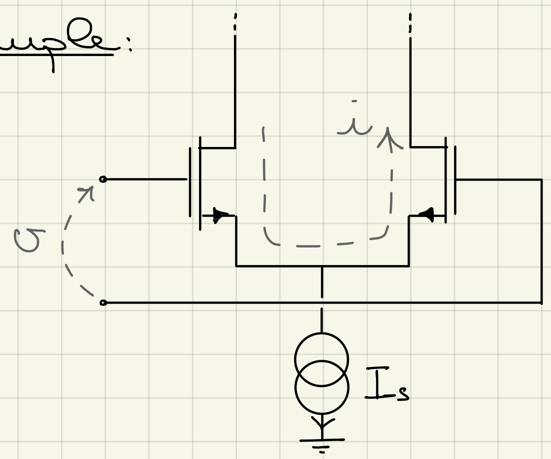
Non-linear G_m

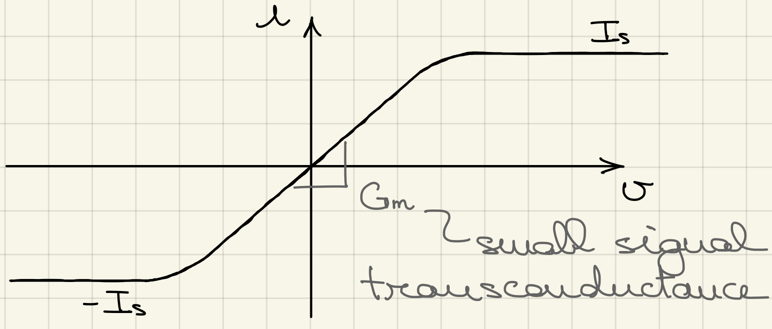


With small signal: $G_m > \frac{1}{R}$ hence oscillator starts up.

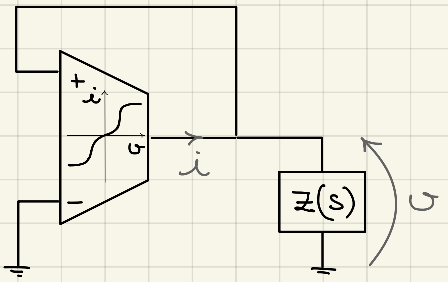
Oscillation then increases until the transconductance saturates.

Example:





We now need to ask ourselves what amplitude of oscillation will the stabilized system settle at.



non-linear function

- $i(t) = i(v(t)) = i\left(\sum_{k=-\infty}^{+\infty} \bar{V}_k e^{jk\omega_0 t}\right) =$

$$= \sum_{k=-\infty}^{+\infty} \bar{I}_k e^{jk\omega_0 t}$$

harmonic components

$$v(t) = \sum_{k=-\infty}^{+\infty} \bar{V}_k e^{jk\omega_0 t}$$

periodic with frequency ω_0

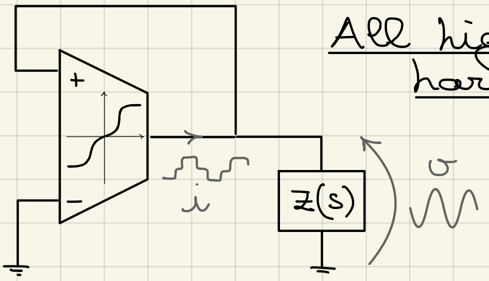
- $\bar{V}_k = \bar{I}_k Z(jk\omega_0)$

$$\begin{cases} \bar{I}_1 \cdot Z(\omega_0) = \bar{V}_1 \\ \bar{I}_2 \; Z(2\omega_0) = \bar{V}_2 \\ \vdots \\ \bar{I}_n \; Z(n\omega_0) = \bar{V}_n \end{cases}$$

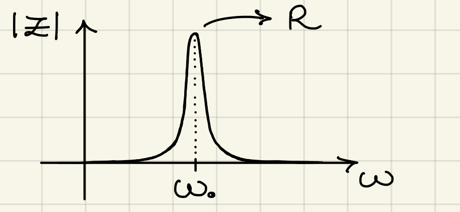
Solve the system to derive the amplitude of each harmonic.

Harmonic Balance (HB) with n-harmonics

In our analysis, however, we assume to study HARMONIC OSCILLATORS ($v(t)$ is a pure sinusoid) i.e with a high Q factor



All higher order harmonics are suppressed



$$\underline{Z(j\omega) \simeq 0 \quad \forall \omega \neq \omega_0}$$

The HB is reduced to: $\bar{I}_1 \, Z(j\omega_0) = \bar{V}_1 \longrightarrow \boxed{Z(j\omega_0) = \frac{\bar{V}_1}{\bar{I}_1}}$

Defining $G_{m_h} := \frac{\bar{I}_1}{\bar{V}_1}$ harmonic (effective) transconductance

amplitudes of 1st harmonics

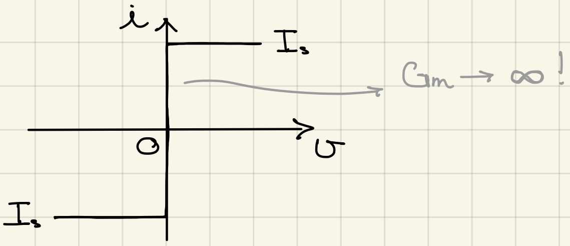
it is $Z(j\omega_0) = \frac{1}{G_{mh}}$

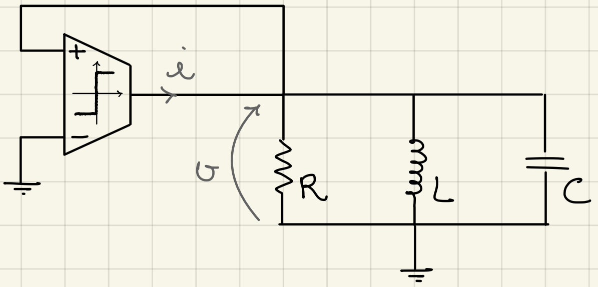
and we can re-write the oscillation condition as

$$\{G_{mh} \cdot Z(j\omega_0) = G_{loop_h}(j\omega_0) = 1\}$$

we replaced the small signal G_m with a harmonic G_m (method of "descriptive function")

Example: $i(v) = I_s \, \mathrm{sign}(v(t))$

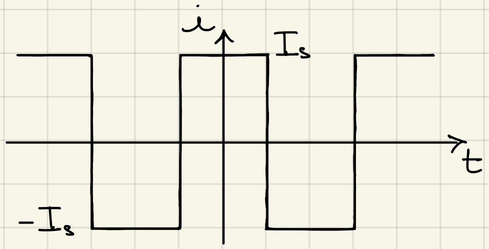




Hp: $v(t) = A_0 \cos \omega_0 t$ (harmonic oscillator)

$\Rightarrow \bar{V}_1 = A_0$

$\Rightarrow$



Oscillator condition:

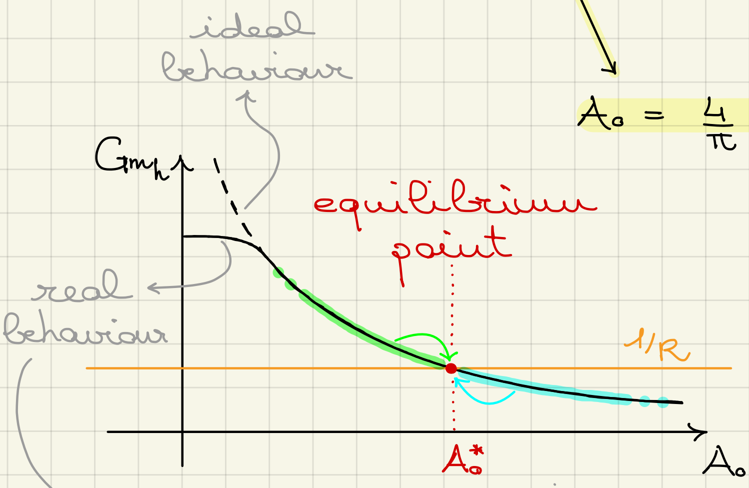
$$G_{loop_h}(j\omega_0) = 1$$

$$\Rightarrow \begin{cases} G_{m_h} \cdot R = 1 \\ \omega_0 = \frac{1}{\sqrt{LC}} \end{cases}$$

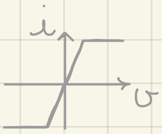
$\Rightarrow \bar{I}_1 = \frac{4}{\pi} I_s$

$\Rightarrow G_{mh} = \frac{4}{\pi} \frac{I_s}{A_0}$

$$A_0 = \frac{4}{\pi} I_s R = A_0^*$$



non-instantaneous transition of G_m



- If $A_0 > A_0^*$: $G_{m_h} \cdot R < 1$ poles in LHP $\Rightarrow A_0$ decreases
- If $A_0 < A_0^*$: $G_{m_h} \cdot R > 1$ poles in RHP $\Rightarrow A_0$ increases

Oscillator design rules:

small signal G_{loop}

1. startup condition $\underline{G_{loop}(j\omega_0) = EG > 1}$

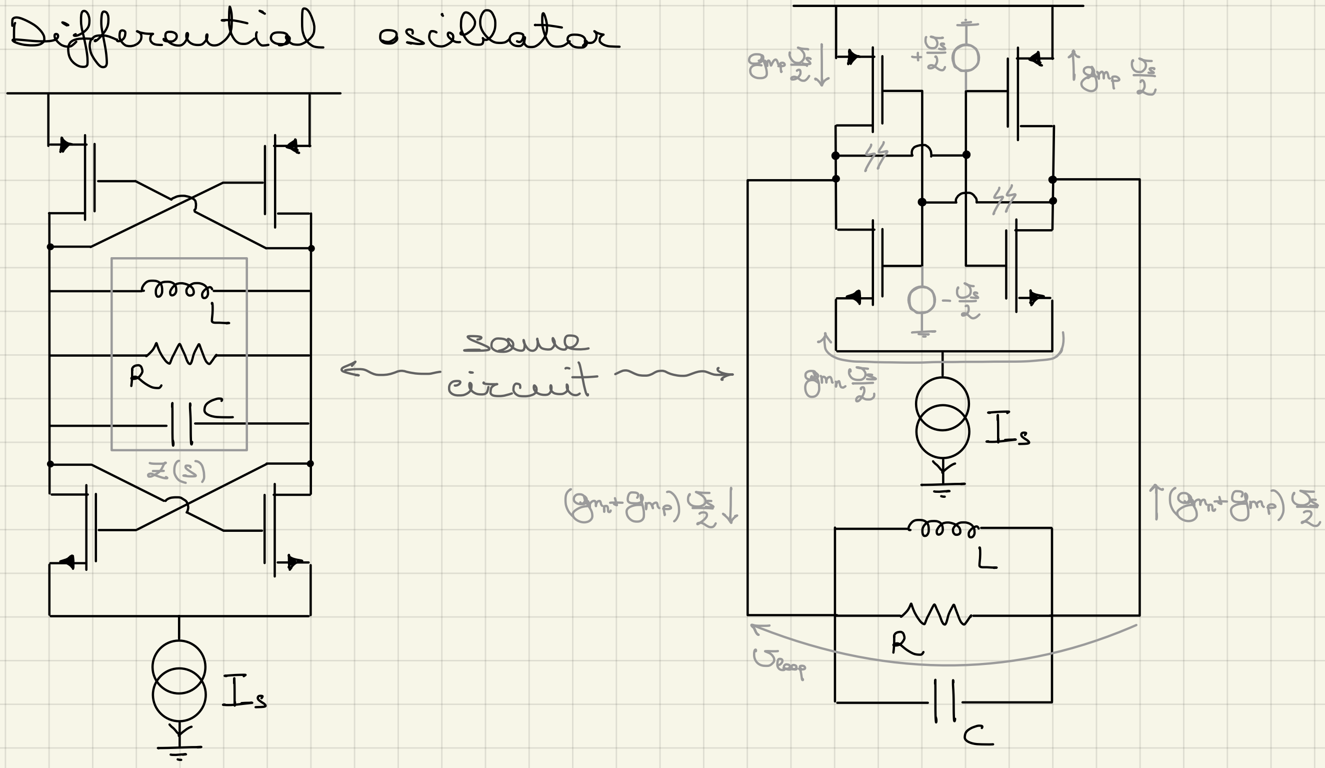
where EG (Excess Gain) is a constant that represents the startup margin (larger EG, faster startup)

harmonic G_{loop}

2. oscillation amplitude $\underline{G_{loop_h}(j\omega_0) = 1}$

Examples of real oscillators:

- Differential oscillator



$$\boxed{G_{loop}(s) = \frac{V_{loop}}{V_s} = Z(s)\cdot\underbrace{\frac{g_{mn}+g_{mp}}{2}}_{\text{small signal } G_m}}$$

(differential loop gain)

Oscillation condition: $G_{loop}(j\omega_0) = 1$

1. $\measuredangle G_{loop}(j\omega_0) = 0$

$\measuredangle Z(j\omega_0) = 0$

$\omega_0 = \omega_z = \frac{1}{\sqrt{LC}}$

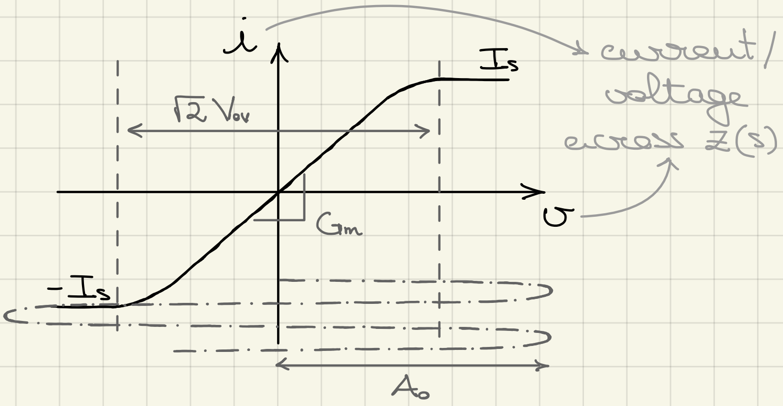
2. $|G_{loop}(j\omega_0)| = 1$

$\frac{g_{mn}+g_{mp}}{2}\cdot R = 1$

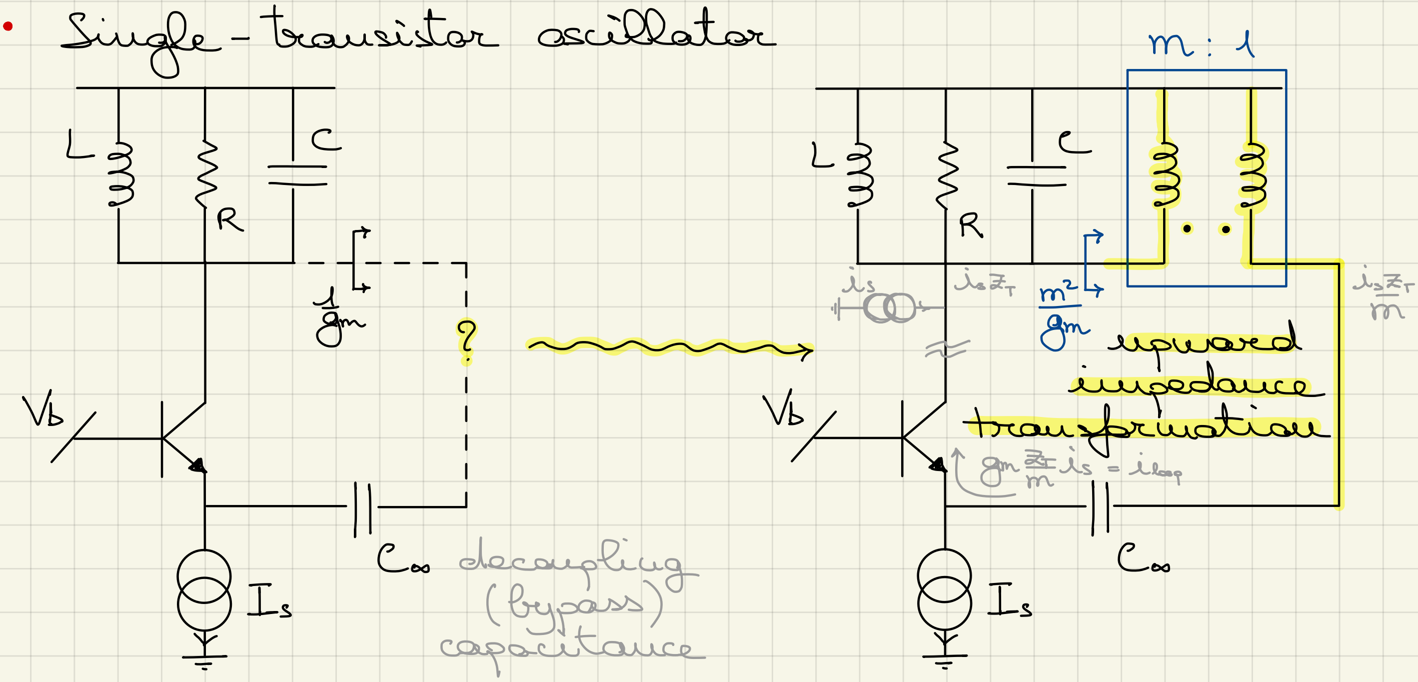
Oscillation amplitude:

assuming $A_0 \gg \sqrt{2}\, V_{ov}$

$\Longrightarrow A_0^* \simeq \frac{4}{\pi} I_s \cdot R$



- Single-transistor oscillator



How can we connect the emitter to the resonator without spoiling the resonator's Q?

$$\left[\, G_{loop}(s) = \frac{I_{loop}}{I_s} = Z_T(s) \cdot \frac{1}{m} \cdot g_m \,\right]$$

where $Z_T(s) = R_T\, H(s)$,

$R_T = \frac{m^2}{g_m} \,/\!/\, R$,

$H(s) = \frac{s\, \omega_r / Q}{s^2 + s\, \omega_r/Q + \omega_r^2}$

Oscillation condition:

$$G_{loop}(j\omega_0) = 1$$

$$\frac{g_m}{m} \cdot R_T\, H(j\omega_0) = 1$$

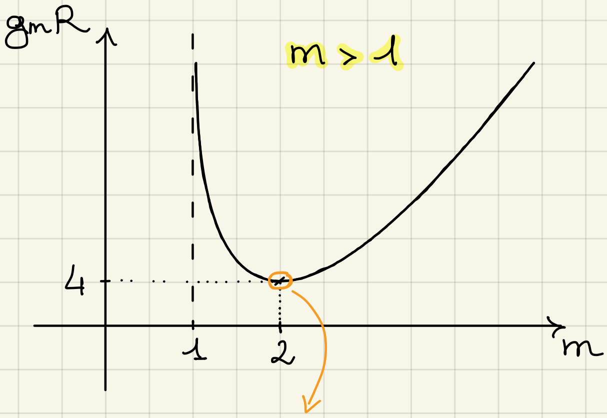
1. $\measuredangle G_{loop}(j\omega_0) = 0$

$\omega_0 = \omega_r$

2. $|G_{loop}(j\omega_0)| = 1$

$\frac{g_m}{m} \cdot R_T = 1$

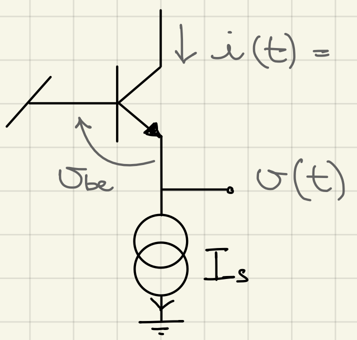
$g_m R = \frac{m}{1 - 1/m}$

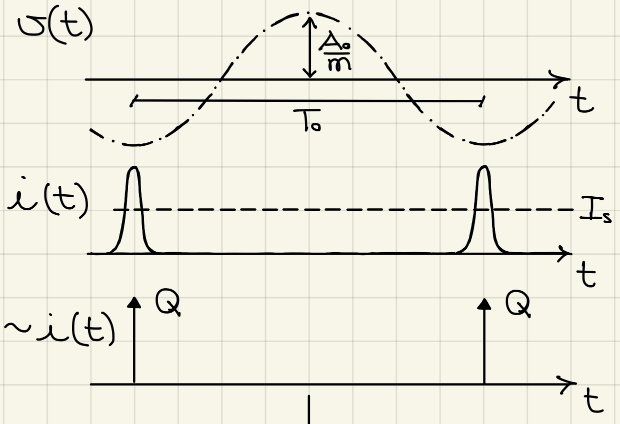


optimum choice for gain

$\Longrightarrow$ If m is too low, losses are too high since R_T is too small.
If m is too high, G_{loop} is attenuated too much.

Oscillation amplitude:





assuming

$$\frac{A_0}{m} >> V_{th} = \frac{KT}{q}$$

$$\rightarrow \bar{I}_1 \simeq 2\frac{Q_0}{T} = 2I_s$$

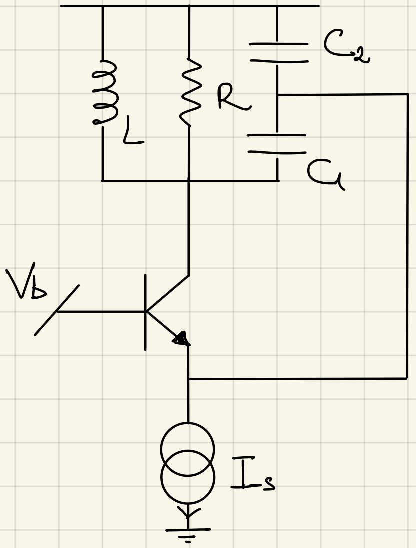
DC component hence all components equal to I_s

$$g_{mh} = \frac{\bar{I}_1}{\bar{V}_1} = \frac{2\,Q_0/T_0}{A_0/m} = \frac{2\,I_s}{A_0/m}$$

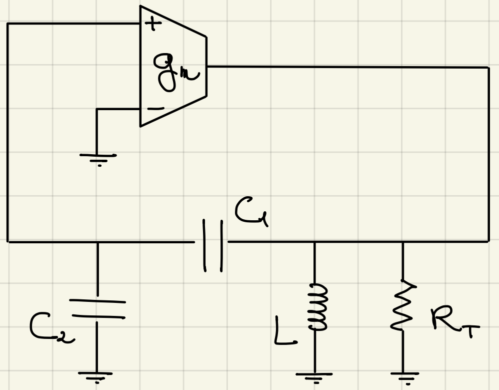
$$G_{looph}(j\omega_0) = 1 \longrightarrow g_{mh} R = \frac{m}{1-1/m}$$

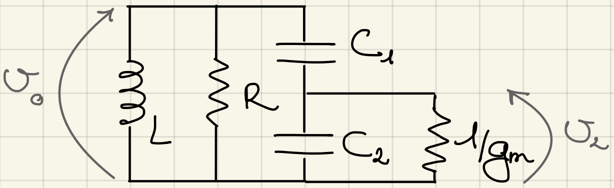
$$\frac{2I_s}{A_0^*}m \cdot R = \frac{m}{1-1/m} \Longrightarrow A_0^* = 2\,I_s R\left(1-\frac{1}{m}\right)$$

- Colpitts oscillator



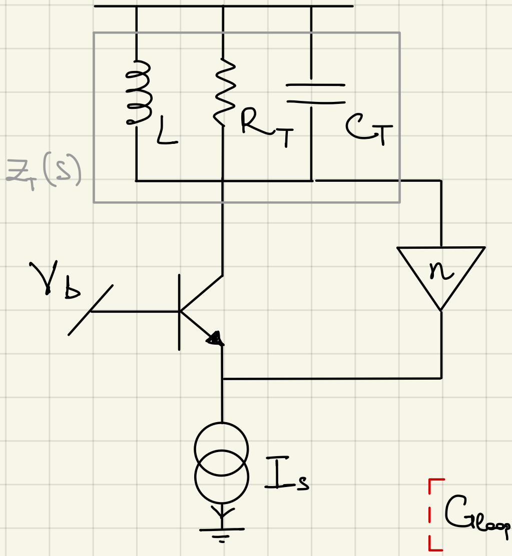
small signal





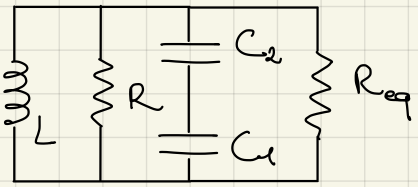
Lossless approx.: $\frac{1}{g_m} >> \frac{1}{\omega_0 C_2} \rightarrow V_2 \simeq \frac{C_1}{C_1+C_2} V_0$

$n<1$



$R_T = R \,//\, R_{eq}$

$C_T = C_1 \,//\, C_2$



$$\Longrightarrow R_{eq} \simeq \frac{1}{g_m}\frac{1}{\left|\frac{V_2}{V_0}\right|^2} = \frac{1}{n^2 g_m}$$

same dissipated power

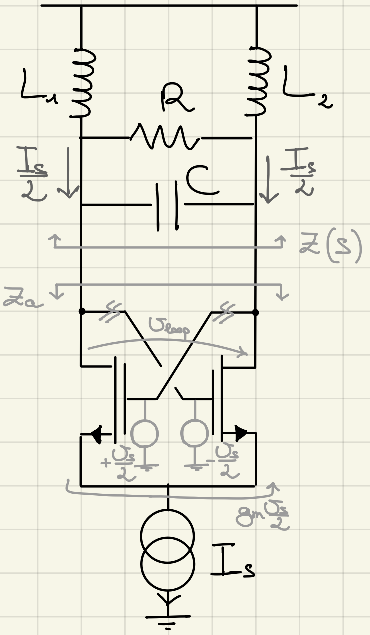
$$\boxed{G_{loop}(s) = Z_T(s)\, n\, g_m}$$

Oscillator condition: (same as single-transistor osc.)

$n \longleftrightarrow \frac{1}{m}$

$$G_{loop}(j\omega_0)=1 \longrightarrow \begin{cases} 1.\ \omega_0=\frac{1}{\sqrt{LC_T}} \\ 2.\ g_m R_T n = 1 \longrightarrow g_m R = \frac{1}{n(1-n)} \end{cases}$$

- Differential oscillator with single transconductor



$$\boxed{G_{loop}(s)=\frac{g_m}{2}Z(s)}$$

Oscillation condition:

$$\begin{cases} 1.\ \omega_0=\frac{1}{\sqrt{LC}} \\ 2.\ \frac{g_m}{2}R=1 \end{cases}$$

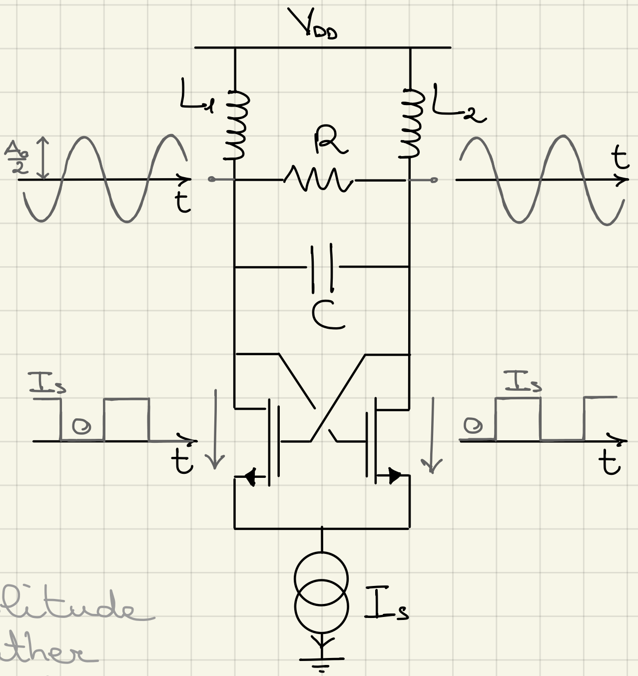
i.e. $Z_a(j\omega_0)=-Z(j\omega_0)$

Oscillation amplitude:

$$|G_{m_h}|R=1$$

$$G_{m_h}=\frac{\bar{I}_1}{\bar{V}_1}=\frac{\frac{2}{\pi}I_s}{A_0}$$

$$\Longrightarrow A_0^*=\frac{2}{\pi}I_sR$$

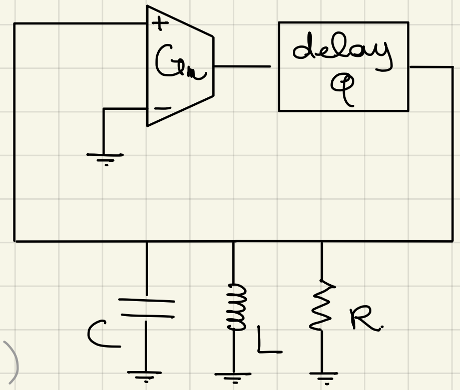


(Note that the maximum amplitude is not limited by V_{DD} but rather by I_s, which needs a certain voltage drop across its terminals to provide the full I_s current; also note that the two MOSFETs are alternating between on and off states, but their exact state when on, either triode or saturation, is not relevant)

Frequency stability

To measure the sensitivity of the oscillation frequency to non-idealities.

E.g.: add an extra delay φ in the loop of a RLC osc.



Oscillation condition:

$$\measuredangle G_{loop}(j\omega_0) = 0$$

(| G_{loop} | = 1 isn't affected)

$$-\varphi + \measuredangle Z(j\omega_0) = 0$$

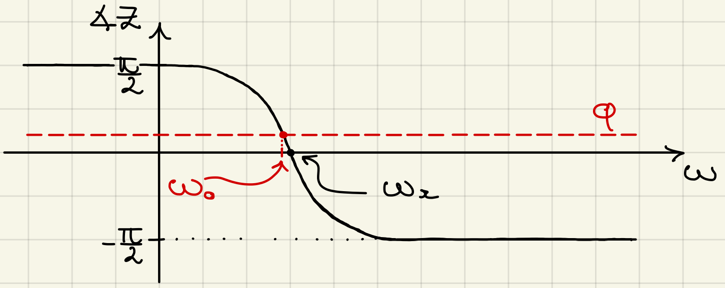
$$-\varphi + \frac{\pi}{2} - \text{arctg}\left(\frac{\omega_0\omega_r/Q}{\omega_r^2-\omega_0^2}\right) = 0$$

$$\xrightarrow{*} \left\{\text{arctg}\left(\frac{\omega_r^2-\omega_0^2}{\omega_0\omega_r/Q}\right) = \varphi\right\}$$

$$G_{loop}(s) = G_m\, e^{-j\varphi}\, Z(s)$$

$$Z(s) = \frac{s\,\omega_r/Q}{s^2 + s\,\omega_r/Q + \omega_r^2} \cdot R$$

$$* \quad \frac{\pi}{2} - \text{arctg}(x) = \text{arctg}\left(\frac{1}{x}\right)$$



$$\Longrightarrow \omega_0 < \omega_r$$

$\uparrow$ (of course $\varphi > 0$ being a delay)

$$\overset{\omega_0-\omega_r}{\overset{\shortparallel}{\Delta\omega_0}} \simeq \frac{\Delta\varphi^{\,=\varphi}}{\left.\frac{d\measuredangle Z}{d\omega_0}\right|_{\omega_0=\omega_r}} = \varphi \cdot \frac{1}{\left[\frac{1}{1+\left(Q\frac{\omega_r^2-\omega_0^2}{\omega_0\omega_r}\right)^2} \cdot \frac{Q}{\omega_r} \cdot \frac{-2\omega_0^2-(\omega_r^2-\omega_0^2)}{\omega_0^2}\right]_{\omega_0=\omega_r}} =$$

$$= -\varphi\,\frac{\omega_r}{2Q}$$

$$\Longrightarrow \left[\frac{\Delta\omega_0}{\omega_0} = -\frac{\Delta\varphi}{2Q}\right]$$

$\updownarrow$

frequency stability

relative frequency variation induced by an extra delay φ is inversely proportional to Q

What about an extra delay in the loop of a ring osc?

neg. f.b.

$$G_{loop}(s) = \frac{G^3}{(1+s\tau)^3}\, e^{-j\varphi} \quad (G>0)$$

$$\measuredangle G_{loop}(j\omega_0) = -\pi \longrightarrow -\varphi - 3\,\text{arctg}(\omega_0\tau) = -\pi \longrightarrow \omega_0 = \frac{\text{tg}\left(\frac{\pi-\varphi}{3}\right)}{\tau}$$

$$\Longrightarrow \omega_0 < \omega_r = \frac{\sqrt{3}}{\tau} = \frac{\text{tg}\,\pi/3}{\tau}$$

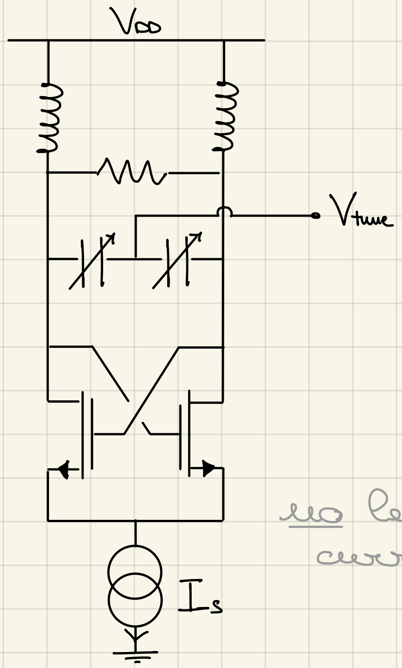
$$\overset{\omega_0-\omega_r}{\overset{\shortparallel}{\Delta\omega_0}} \simeq \frac{\Delta\varphi}{\frac{d}{d\omega_0}\left[-3\,\text{arctg}(\omega_0\tau)\right]_{\omega_0=\omega_r}} = \varphi \cdot \frac{1}{-3\tau\left[\frac{1}{1+(\omega_0\tau)^2}\right]_{\omega_0=\omega_r}} = -\varphi\,\frac{1}{\frac{3\tau}{4}} =$$

$$= -\varphi \frac{4}{3\sqrt{3}} \omega_r$$

$$\rightarrow \left[\frac{\Delta\omega_0}{\omega_0} = -\frac{4}{3\sqrt{3}} \Delta\varphi \right]$$

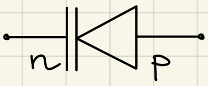
Voltage-Controlled Oscillators (VCOs)

→ use of variable capacitors (varactors)



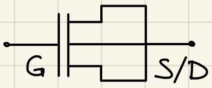
2 main options:

a) p-n junction — has leakage current!



in reverse biasing: $C = \frac{C_0}{\left(1+\frac{V}{V_0}\right)^m}$

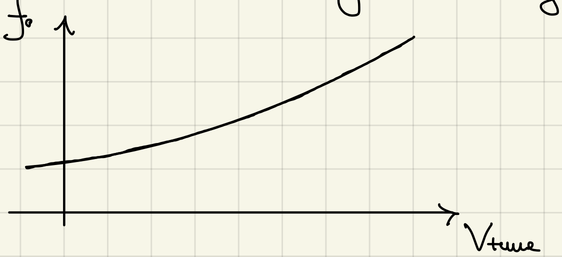
b) MOS junctions — no leakage current



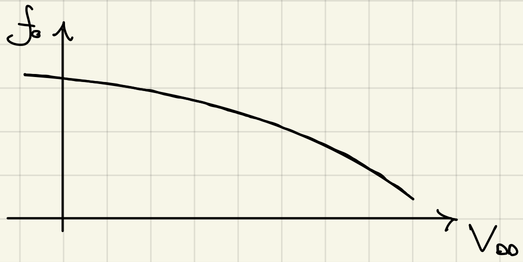
- from inversion to depletion
- from accumulation to depletion

Phase noise

- Indirect: AM-to-FM conversion i.e. upconversion of low frequency noise



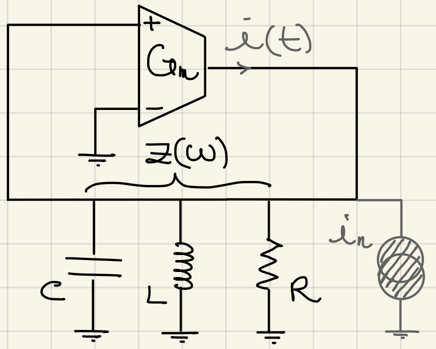
$K_{vco} = 2\pi \frac{\partial f_0}{\partial V_{tune}}$ VCO sensitivity (or gain)



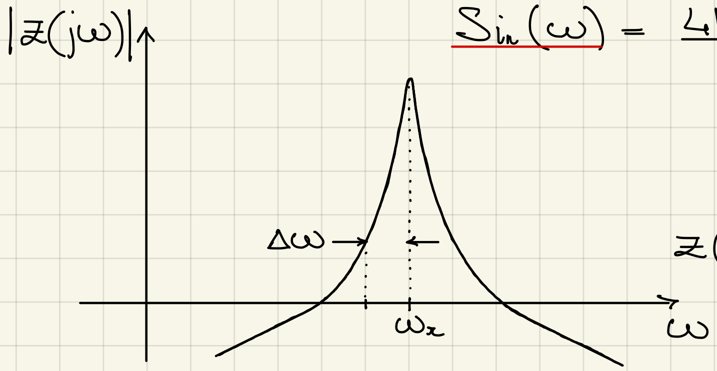
$K_{V_{DD}} = 2\pi \frac{\partial f_0}{\partial V_{DD}}$ VCO supply pushing

$$S_{V_{tune}}(\omega) \xrightarrow[FM]{} S_{\omega_0}(\omega) = K_{vco}^2 S_{V_{tune}}(\omega) \xrightarrow[PM]{} S_\varphi(\omega) = \frac{S_{\omega_0}(\omega)}{\omega^2} = \frac{K_{vco}^2}{\omega^2} S_{V_{tune}}(\omega)$$

- Direct: $i_n(t)$ is noise associated to tank losses (resistor R)



$$S_{in}(\omega) = \frac{4KT}{R}$$



$$\omega = \omega_r \pm \Delta\omega$$

$$Z(j\omega) = R \cdot \frac{j(\omega_r \pm \Delta\omega)\omega_r/Q}{\omega_r^2 - (\omega_r \pm \Delta\omega)^2 + j(\omega_r \pm \Delta\omega)\frac{\omega_r}{Q}} = R \frac{1}{1 + j\frac{\Delta\omega}{\frac{\omega_r}{Q}} \cdot \frac{\Delta\omega \pm 2\omega_r}{\omega_r \pm \Delta\omega}}$$

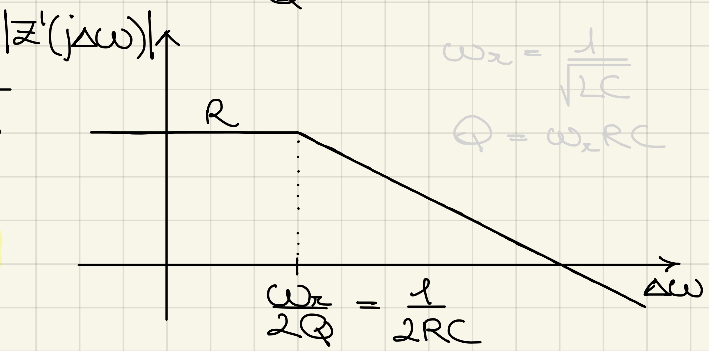
$\boxed{\Delta\omega << \omega_r}$

$$\Longrightarrow Z(j\omega_r \pm j\Delta\omega) \simeq \frac{R}{1 \pm j\frac{\Delta\omega}{\omega_r} \cdot 2Q}$$

$$Z'(\pm j\Delta\omega) := \frac{R}{1 \pm j2RC\Delta\omega}$$

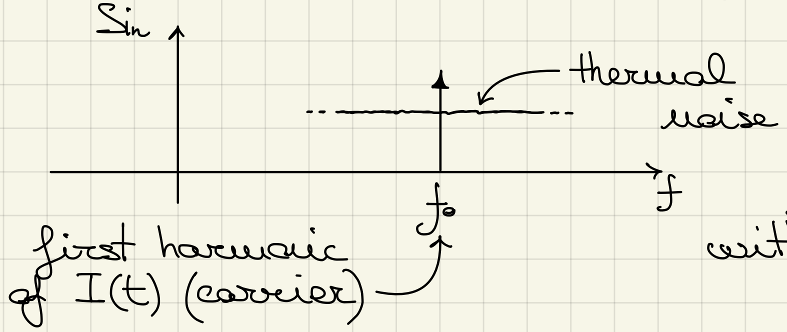
baseband equivalent of $Z(j\omega)$ of RLC resonator around resonance

frequency offset from carrier



$\omega_r = \frac{1}{\sqrt{LC}}$

$Q = \omega_r RC$



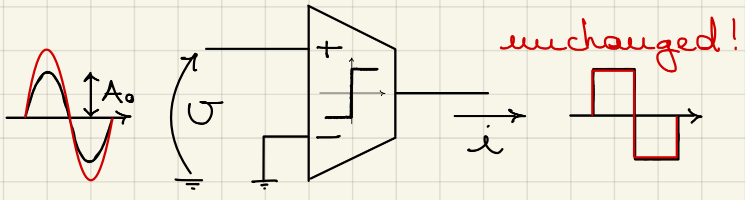
$S_{in} = \frac{4KT}{R}$ white noise

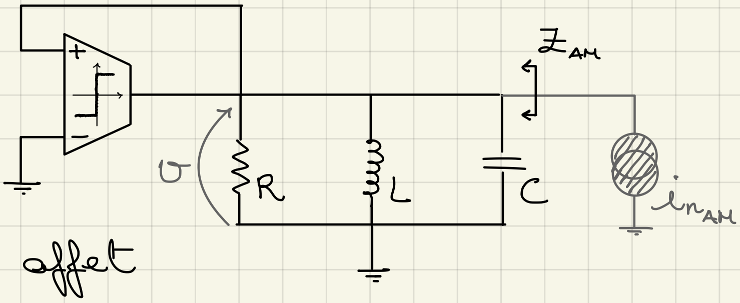
Rice theorem

in phase with carrier $S_{in_{AM}} = \frac{2KT}{R}$

in quadrature with carrier $S_{in_{PM}} = \frac{2KT}{R}$

- AM noise component:

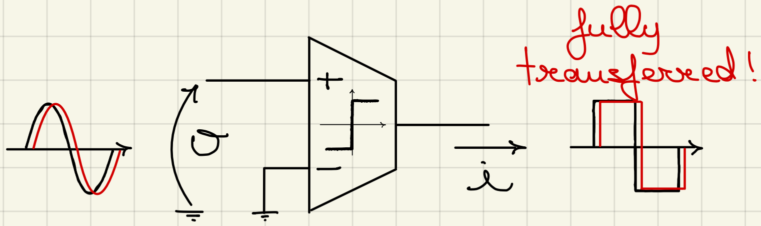




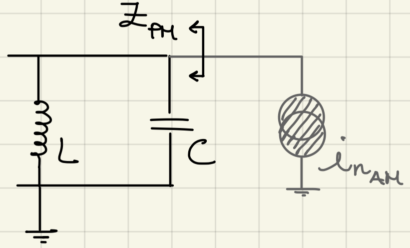
Amplitude noise doesn't affect transconductor operation.

$\Longrightarrow Z_{AM}(\omega) = Z(\omega)$ i.e. without considering the OTA

- PM noise component:

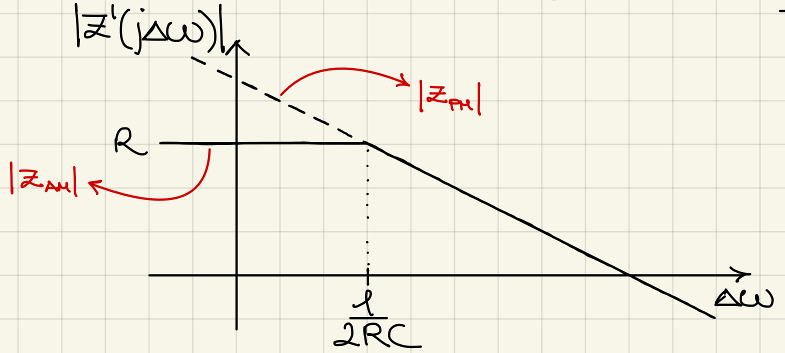


Transconductor fully compensates current injected in R.



$$\Longrightarrow Z_{PM}(\omega) = Z(\omega)\Big|_{R\to+\infty}$$

i.e. considering $Z_a = -\frac{1}{G_m} = -R$



$$\Longrightarrow S_v(\omega) = \frac{2KT}{R}|Z_{AM}(\omega)|^2 + \frac{2KT}{R}|Z_{PM}(\omega)|^2$$

$\boxed{\Delta\omega \ll \omega_r}$: $Z_{AM}(j\omega_r \pm j\Delta\omega) \simeq R$ $\qquad Z_{PM}(j\omega_r \pm j\Delta\omega) \simeq \frac{1}{\pm j2\Delta\omega C}$

$$S_v(\omega_r \pm \Delta\omega) = \overset{AM}{\frac{2KT}{R}R^2} + \overset{PM}{\frac{2KT}{R}\frac{1}{4\Delta\omega^2C^2}} \frac{R\omega_r^2}{R\omega_r^2}$$

$$= 2KTR + \frac{1}{2}KTR\left(\frac{\omega_r}{Q}\right)^2\frac{1}{\Delta\omega^2}$$

dominant term

$$\Longrightarrow \mathcal{L}(\Delta\omega) := \frac{S_v(\omega_r+\Delta\omega)}{P_{carrier}} \simeq \frac{\cancel{\frac{1}{2}}KTR\cdot\left(\frac{\omega_r}{Q}\right)^2\cdot\frac{1}{\Delta\omega^2}\cdot F_a}{\frac{A_0^2}{\cancel{2}}} =$$

F_a → correction factor $F_a \geq 1$ to include noise due to active elements

$$P_R = \frac{A_0^2}{2R}$$

$$= \frac{KT}{2P_R}\left(\frac{\omega_r}{Q}\right)^2\frac{1}{\Delta\omega^2}\cdot F_a$$

resonator efficiency $0 \leq \eta \leq 1$

$$P_R = \eta P_{DC}$$

power dissipated in the resonator

DC power from supply

$$\Longrightarrow \left[\mathcal{L}(\Delta\omega) \simeq \frac{KT}{2\eta P_{DC}}\left(\frac{\omega_r}{Q}\right)^2\cdot\frac{1}{\Delta\omega^2}F_a\right]$$

trade-off between phase noise dnd dissipated power

Let us define a Figure of Merit for oscillators:

$$FoM_{dB} := 10\log_{10}\left\{\frac{1}{\mathcal{L}(\Delta\omega)P_{DC,mW}}\left(\frac{\omega_{osc}}{\Delta\omega}\right)^2\right\}$$

ω_r

$$= 10\log_{10}\left\{\frac{(\cancel{\omega_{osc}}/\cancel{\Delta\omega})^2}{\frac{KT}{2\eta P_{DC}}\left(\frac{\cancel{\omega_{osc}}}{Q}\right)^2\frac{1}{\cancel{\Delta\omega}^2}F_a P_{DC,mW}}\right\}$$

$$= 10\log_{10}\left\{10^{-3}\cdot\frac{2\eta}{KT}Q^2\frac{1}{F_a}\right\}$$

Thermodynamic limit of FoM of oscillators:

ideally $\eta = 1 \Longrightarrow \mathrm{FoM}_{dB_{max}} = 10\log_{10}\left\{\frac{2}{KT}\frac{Q^2}{F_a}\right\} - 30\mathrm{dB}$

$= 197\,\mathrm{dB}$ for $Q = 10$, $F_a = 1$

e.g.: $f_{osc} = 1\mathrm{GHz}$

$\Delta f = 1\mathrm{MHz}$

$P_{DC} = 1\mathrm{mW}$

$Q = 10$

by def. of FoM

$$\Longrightarrow \mathcal{L}_{min}(\Delta f) \overset{\downarrow}{=} \frac{1}{\mathrm{FoM}_{max}} \cdot \frac{1}{P_{DC,min}} \cdot \left(\frac{f_{osc}}{\Delta f}\right)^2 =$$

$$= -\mathrm{FoM}_{dB_{max}} - 10\log_{10} P_{DC,min} + 20\log_{10}\left(\frac{f_{osc}}{\Delta f}\right) =$$

$$= -197\,\mathrm{dB} - 0\,\mathrm{dBm} + 60\,\mathrm{dB} = -137\,\frac{\mathrm{dBc}}{\mathrm{Hz}}$$

Circuit simulators (e.g. Cadence Spectre, Mentor Eldo, ...)

• DC	DC analysis	bias point (non-linear)
• AC	AC analysis	transfer functions (linear*)
• NOISE	noise analysis	based on AC (linear*)

← LTI approximation
Linear Time-Invariant

*non-linear devices are replaced by equivalent linear circuits

• TRAN	transient analysis does not account for noise	transient behaviour (non-linear)
• NOISETRAN	transient analysis with noise	noise source are modelled as random sequences, needs many runs to get statistics

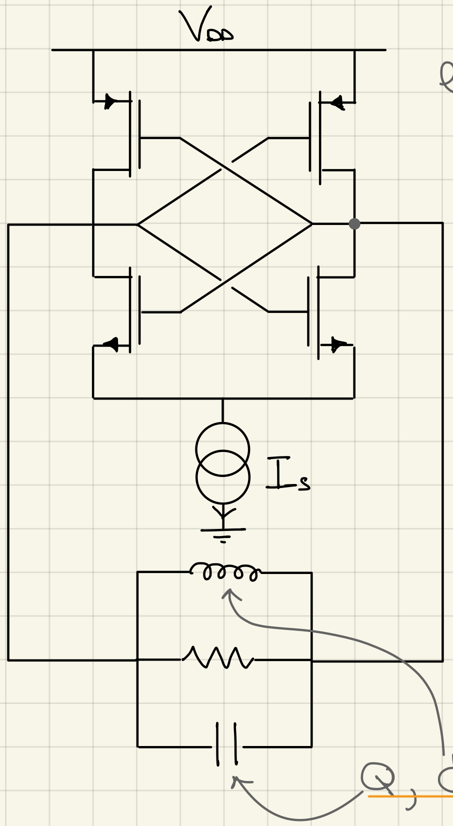
↓ time consuming

sub-group of circuit simulators
↑

RF circuit simulators (e.g Spectre RF, Eldo RF, ...)

- PSS periodic steady state analysis (non-linear)

  searches for T_0 (period of oscillation) that satisfies a periodic steady state



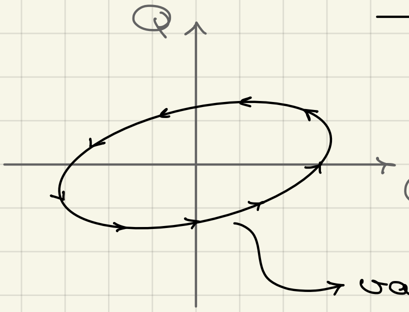
$v_1(t+T_0) = v_1(t)$

$v_2(t+T_0) = v_2(t)$

⋮

for every voltage and current

→ find T_0



after T_0, the state variables (and therefore every other variable of the system) will be back where they started

Q, Φ state variables

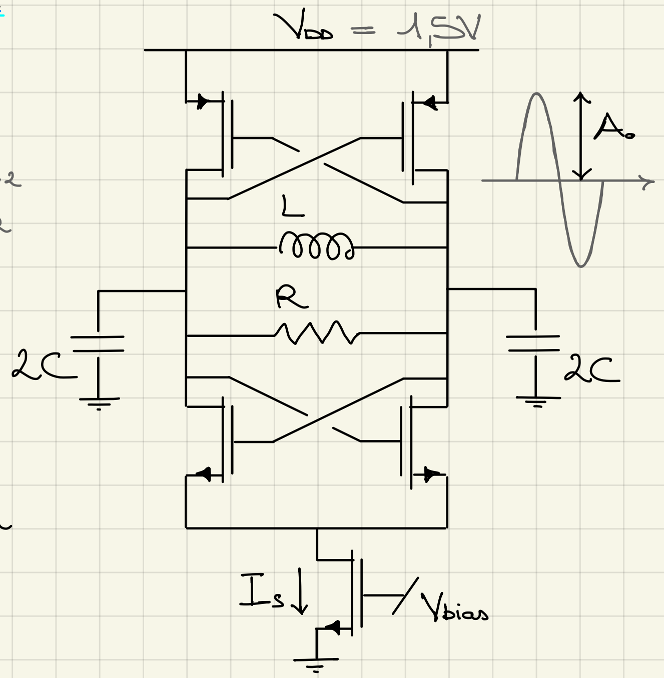
- **PAC** periodic AC analysis — transfer functions of small signal perturbations (linear*)

→ LTV approx
Linear Time-Variant

- **PNOISE** periodic noise analysis (linear*)

* linearization occurs around a bias point that is not constant but periodic

Design of an LC oscillator



MOS devices:
- $|V_T| = 0{,}35V$
- $|\mu_n C'_{ox}| = 120\,\mu A/V^2$
- $|\mu_p C'_{ox}| = 60\,\mu A/V^2$

Specifications:
- $f_0 = 1{,}5GHz$
- $Q = 20$
- $I_s = 3mA$
- max FoM

Unknowns:
- R
- L
- C
- A_0
- $\mathcal{L}(\Delta f)$ ↓ 1MHz
- $(W/L)_n$
- $(W/L)_p$

1. Startup: $G_{loop}(j\omega_0) = EG$

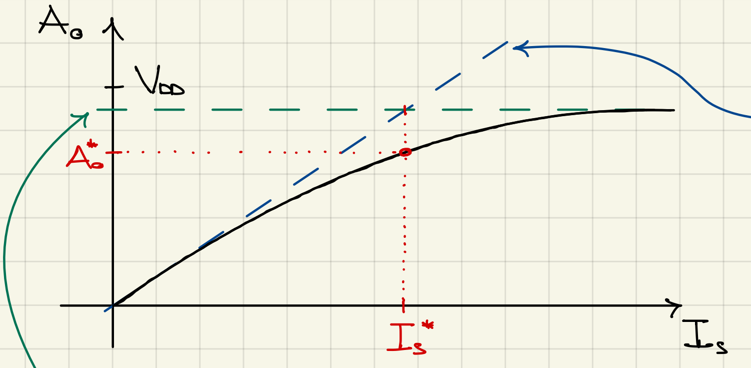
$G_m R = EG > 1$ e.g. $EG = 5$

2. Maximize FoM $\propto \frac{2\eta}{KT} \cdot Q^2 \longrightarrow$ maximize η

$$\eta = \frac{P_R}{P_{DC}} = \frac{A_0^2/2R}{I_s \cdot V_D} \longrightarrow \text{maximize } A_0$$

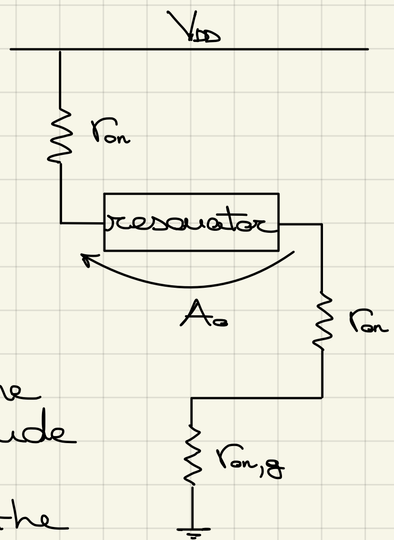
Oscillation amplitude: $G_{m_h} \cdot R = 1$

$$\frac{4}{\pi}\frac{I_s}{A_0} \cdot R = 1$$



$A_0 = \frac{4}{\pi} I_s R$ (current-limited regime)

(voltage-limited regime) $A_0 \simeq V_{DD} \frac{R}{R + 2r_{on} + r_{on,g}}$



⟶ Optimize I_s to achieve the largest oscillation amplitude with the lowest current consumption (i.e where the two regimes cross each other)

3. In a spreadsheet:

Data

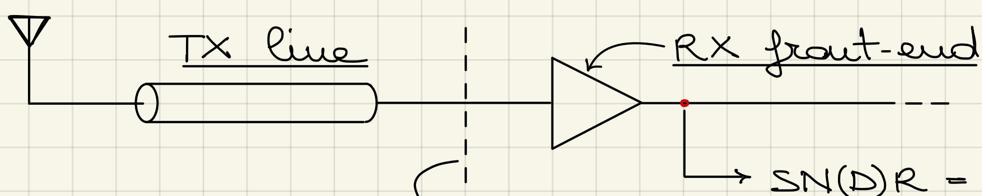
f_0	1,5GHz
Q	20
V_{DD}	1,5V
$\mu C'_{ox}$	$60/120 \frac{\mu A}{V^2}$
V_t	0,35 V
I_s	3mA
EG	5
A_0	0,9V

Equations

$R = \frac{\pi}{4}\frac{A_0}{I_s}$	236Ω
$L = \frac{R}{\omega_0 Q}$	1,25nH
$C = \frac{1}{\omega_0^2 L}$	9pF
$G_m = EG \frac{1}{R}$	$2{,}12 \frac{mA}{V}$
$\left(\frac{W}{L}\right) = \frac{g_m^2}{\mu C'_{ox} I_s}$	2500/1250
$\mathcal{L}(\Delta f) = 10\log_{10}\left\{\frac{KTR}{A_0^2} \cdot \frac{f_0^2}{Q^2} \cdot \frac{1}{\Delta f^2}\right\}$	$-142 \frac{dBc}{Hz}$

$$G_m = \frac{g_{mn} + g_{mp}}{2} \underset{\substack{\uparrow \\ g_{mn} = g_{mp}}}{=} g_m = \sqrt{2\mu C'_{ox}\left(\frac{W}{L}\right)\frac{I_s}{2}}$$

Basics of RF systems

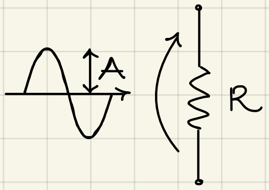


$\rightarrow SN(D)R = \frac{P_s}{P_n + P_d}$

Sensitivity: minimum detectable signal (SNR_{min})

$\hookrightarrow$ limited by:
1. non-linearity
2. impedance matching
3. noise

Note: power of a signal



$$P = \frac{A^2}{2R} = \frac{A_{rms}^2}{R} \quad [W]$$

$$v(t) = A \sin(\omega t)$$

Effects of non-linearity

$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$ $\qquad y(t) = x(t) * h(t)$ ← impulse response

$x(t) \longrightarrow \boxed{LTV} \longrightarrow y(t)$ $\qquad y(t) = x(t) * h(t, \tau)$

$x(t) \longrightarrow \boxed{\text{Non-Linear}} \longrightarrow y(t)$

- memoryless or static model
  $y(t) =$ Taylor series
  $= \alpha_1 x(t) + \alpha_2 x^2(t) + \alpha_3 x^3(t) + \dots$

we suppose, for simplicity, all non-linear systems to be static

- non-linear dynamic system
  $y(t) =$ Volterra series

① Single tone at input

a. Harmonic generation

$x(t) = A \cos \omega t \qquad y(t) \simeq \alpha_1 x(t) + \alpha_2 x^2(t) + \alpha_3 x^3(t)$

$$\left\{ \cos^2 x = \frac{1 + \cos(2x)}{2} \qquad \cos^3 x = \frac{3}{4} \cos x + \frac{1}{4} \cos^3(3x) \right\}$$

$$x^2(t) = \underbrace{\frac{A^2}{2}}_{\text{DC}} + \frac{A^2}{2}\underbrace{\cos(2\omega t)}_{\text{2nd harmonic}} \qquad x^3(t) = \frac{3}{4}A^3\underbrace{\cos\omega t}_{\text{fundamental}} + \frac{A^3}{4}\underbrace{\cos^3(3\omega t)}_{\text{3rd harmonic}}$$

"rectification"

can alter the bias point!

$$y(t) = \underbrace{\alpha_1}_{\text{small signal gain}} A\cos\omega t + \alpha_2\frac{A^2}{2} + \alpha_2\frac{A^2}{2}\cos(2\omega t) +$$

$$+ \alpha_3\frac{3}{4}A^3\cos\omega t + \alpha_3\frac{A^3}{4}\cos(3\omega t)$$

$$= B_0 + B_1\cos\omega t + B_2\cos(2\omega t) + B_3\cos(3\omega t)$$

where $B_0 = \alpha_2\frac{A^2}{2}$ $\quad B_1 = \underbrace{\boxed{\alpha_1 A}}_{\text{desired component}} + \underbrace{\boxed{\alpha_3\frac{3}{4}A^3}}_{\text{unwanted component}}$ *

$B_2 = \alpha_2\frac{A^2}{2}$ $\quad B_3 = \alpha_3\frac{A^3}{4}$

$\Longrightarrow$ - Generated harmonic amplitude:

$$B_n \propto A^n \qquad n \geq 1$$

(nth harmonic has amplitude $\propto A^n$)

- $B_{2n} = 0$ if $\alpha_{2n} = 0 \longleftrightarrow$ fully differential

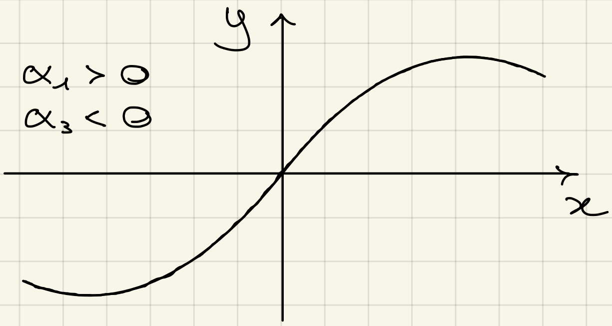
(even-order harmonics come from even-order non-linearities)

b. Gain compression *

$B_1 = \alpha_1 A + \frac{3}{4}\alpha_3 A^3 \longrightarrow$ gain of the system:

it's actually a harmonic gain

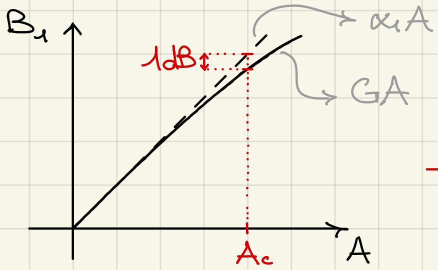
$$G = \frac{B_1}{A} = \alpha_1 + \underbrace{\alpha_3\frac{3}{4}A^2}_{\text{gain compression}}$$



COMPRESSIVE system: $\boxed{\alpha_1\alpha_3 < 0}$

Def (1dB compression point): input amplitude (power) A_c such that the system gain is reduced by 1dB

$$\frac{\text{compress. output ampl.}}{\text{ideal (linear) output ampl.}} = \frac{\alpha_1 A_c + 3/4\,\alpha_3 A_c^3}{\alpha_1 A_c} = 10^{-1/20} = -1\text{dB}$$



$$1 + \frac{3}{4}\frac{\alpha_3}{\alpha_1}A_c^2 = 0{,}89$$

$$\rightarrow A_{c_{dB}} = 20\log_{10}A_c = -9{,}6\,dB + 10\log_{10}\left(\frac{4}{3}\left|\frac{\alpha_1}{\alpha_3}\right|\right)$$

② Two tones at input

$$x(t) = A_1\cos\omega_1 t + A_2\cos\omega_2 t$$

IM3: third-order intermodulation products

$$y(t) = \alpha_1 x(t) + \alpha_3 x^3(t) \quad \text{for simplicity.}$$

$$(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$$

$$y(t) = B_1\cos\omega_1 t + B_2\cos\omega_2 t + B_{221}\cos(2\omega_2-\omega_1)t +$$

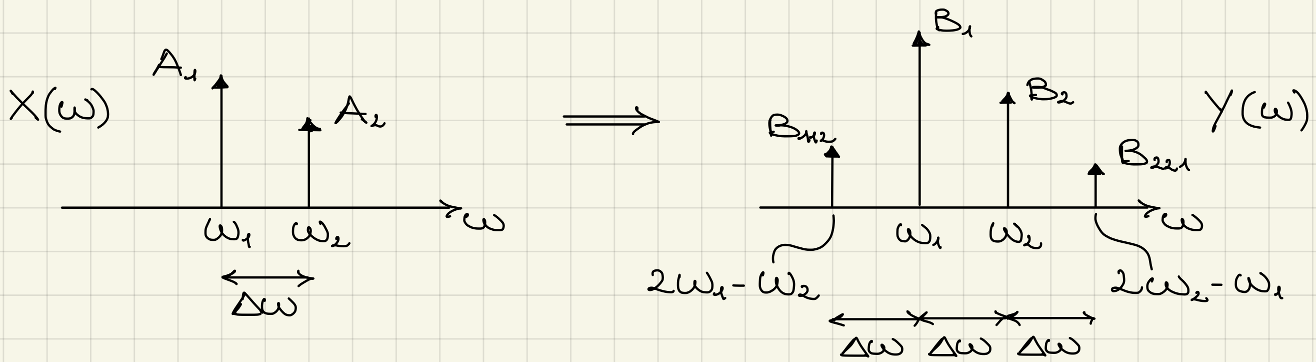
$$+ B_{112}\cos(2\omega_1-\omega_2)t + \ldots$$

where

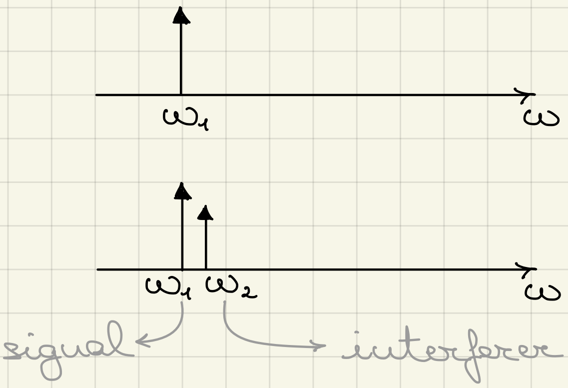
$$B_1 = \alpha_1 A_1 + \frac{3}{4}\alpha_3 A_1^3 + \frac{3}{2}\alpha_3 A_1 A_2^2$$

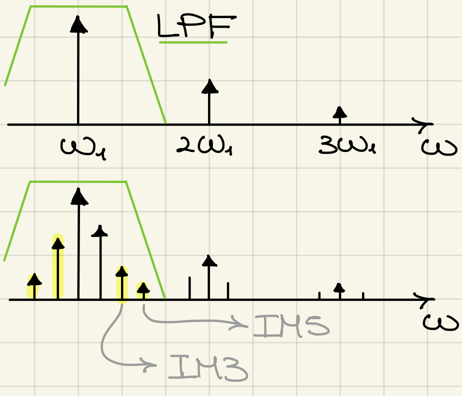
$$B_2 = \alpha_1 A_2 + \frac{3}{4}\alpha_3 A_2^3 + \frac{3}{2}\alpha_3 A_1^2 A_2$$

$$\boxed{\begin{aligned} B_{221} &= \frac{3}{4}\alpha_3 A_1 A_2^2 \\ B_{112} &= \frac{3}{4}\alpha_3 A_1^2 A_2 \end{aligned}}$$



Harmonic generation is not much of a problem in RF systems since higher order harmonics can be easily filtered out.
However, non-linearities also cause intermodulation between the signal and nearby interferers, which cannot be filtered.





a. Blocking

called Blocker

In case of small wanted A_1, large unwanted A_2.

$$B_1 = \alpha_1 A_1 + \frac{3}{4}\alpha_3 A_1^3 + \frac{3}{2}\alpha_3 A_1 A_2^2 \simeq \left(\alpha_1 + \frac{3}{2}\alpha_3 A_2^2\right) A_1$$

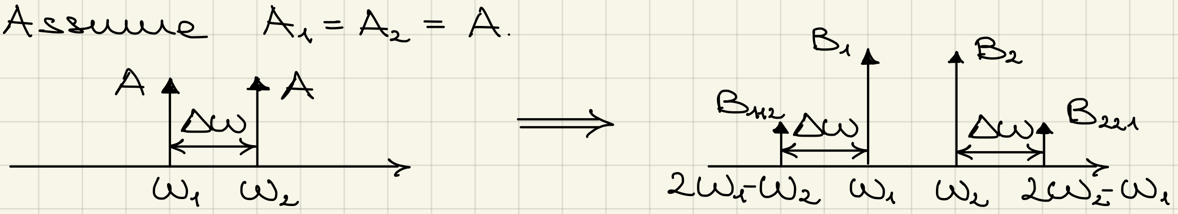
output component at ω_1

negligible if $A_1^3 \ll A_1 A_2^2$

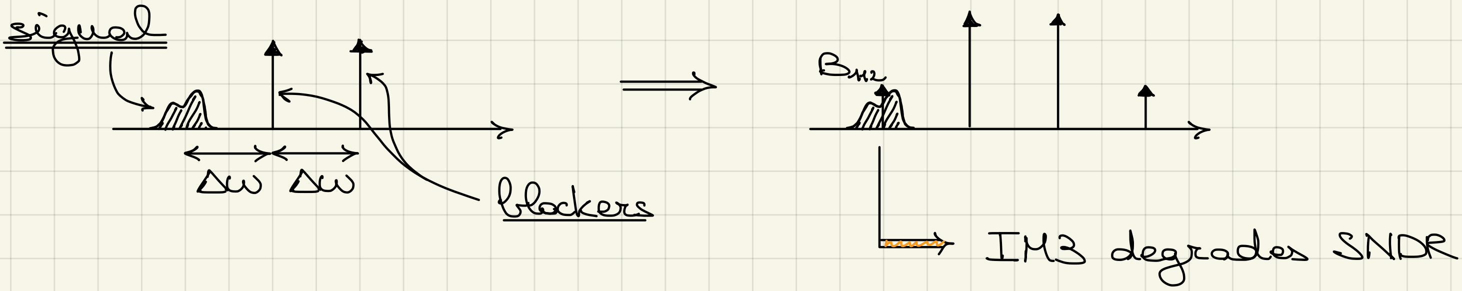
$\rightarrow$ Gain of the system: $G = \dfrac{B_1}{A_1} = \alpha_1 + \dfrac{3}{2}\alpha_3 A_2^2$

b. Intermodulation

Assume $A_1 = A_2 = A$.



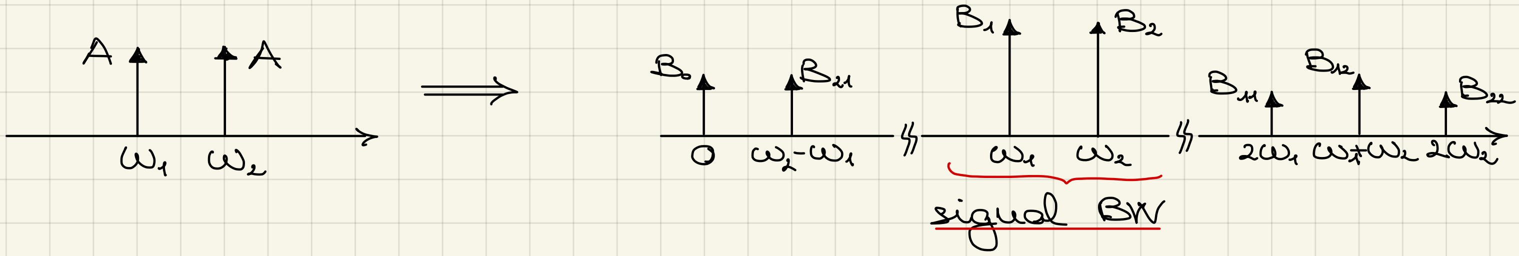
$$B_{221} = B_{112} = \frac{3}{4}\alpha_3 A^3$$



If signal were at $2\Delta\omega$ distance, then IM5 would degrade SNDR, at $3\Delta\omega$ it would be IM7 and so on.

What about second-order non-linearity?

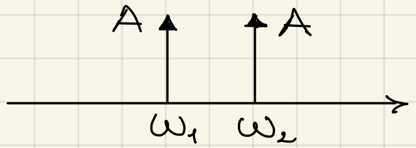
$$\alpha_2 x^2(t) = \alpha_2 A^2 (\cos\omega_1 t + \cos\omega_2 t)^2 \longrightarrow B_0 = B_{12} = B_{21} = 2B_{11} = 2B_{22} = \alpha_2 A^2$$



IM2 products fall outside signal bandwidth.

Introduce now the notion of Intercept Point

E.g. 3rd order IP (IIP3)



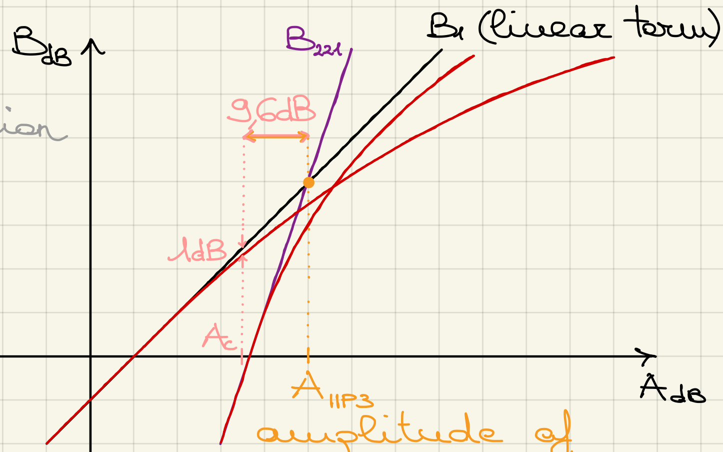
$$B_1 = \alpha_1 A + \frac{3}{4}\alpha_3 A^3 + \frac{3}{2}\alpha_3 A^3 = \alpha_1 A + \frac{9}{4}\underline{\alpha_3 A^3}$$

$$B_{221} = \frac{3}{4}\alpha_3 A^3 + \underline{k.\alpha_5 A^5}$$

intermodulation terms

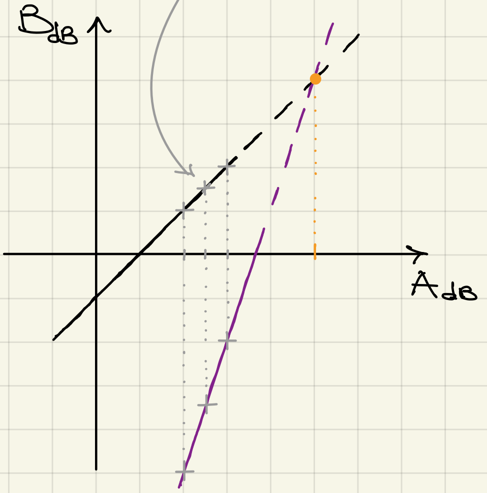
$$20\log_{10}(\alpha_1 A_1) = \\ = \alpha_{1dB} + A_{dB}$$

$$20\log_{10}(B_{221}) = \\ = 20\log_{10}\left(\frac{3}{4}\alpha_3\right) + 3A_{dB}$$



amplitude of input-referred 3rd order intercept point

B_1 and B_{221} both undergo compression due to higher odd-order terms (3rd and 5th, respectively).
Therefore, the IP is typically extrapolated by measuring the response of the system for low A.



"two-tone test"

$$\alpha_1 A_{IIP3} = \frac{3}{4}\alpha_3 A^3_{IIP3} \quad \text{(extrapolated)}$$

$$\Downarrow$$

$$A_{IIP3} = \sqrt{\frac{4}{3}\left|\frac{\alpha_1}{\alpha_3}\right|}$$

$$\downarrow$$

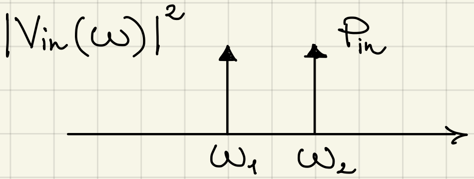
$$\rightarrow \boxed{A_{IIP3_{dB}} = 20\log_{10} A_{IIP3} = 10\log_{10}\left(\frac{4}{3}\left|\frac{\alpha_1}{\alpha_3}\right|\right)}$$

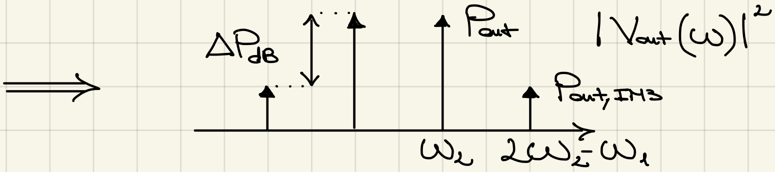
Remember the 1dB compression point:

$$A_{c_{dB}} = -9,6dB + 10\log_{10}\left(\frac{4}{3}\left|\frac{\alpha_1}{\alpha_3}\right|\right)$$

⟹ 1dB compression point is typically about 9,6dB lower than the IIP3

To retrieve A_{IIP3} we actually just need one measurement since the slope is fixed. In fact:

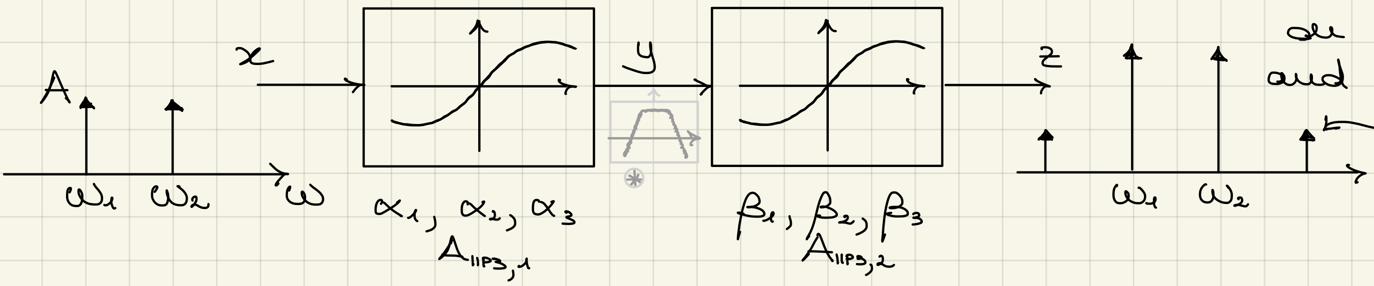




$$\frac{V_{out}(\omega_1)}{V_{out}(\omega_2+\Delta\omega)} = \frac{\overset{B_2}{B_1}}{\underset{B_{112}}{B_{221}}} = \frac{\alpha_1 \overset{|V_{in}|}{A}}{\frac{3}{4}\alpha_3 A^3} = \frac{A^2_{IIP3}}{A^2} \longrightarrow A_{IIP3} = A\sqrt{\frac{V_{out}(\omega_1)}{V_{out}(\omega_2+\Delta\omega)}}$$

$$\Longrightarrow \left[P_{IIP3_{dBm}} = \underbrace{P_{in_{dBm}}}_{\text{input power}} + \frac{1}{2}\underbrace{\Delta P_{dBm}}_{\text{power difference (in dB) between fundamental and IM3}} \right]$$

IIP3 of cascaded stages



will depend on $\alpha_1 \beta_3$ and $\beta_1 \alpha_3$

$$x = A\cos\omega_1 t + A\cos\omega_2 t$$

$$y = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 \qquad z = \beta_1 y + \beta_2 y^2 + \beta_3 y^3$$

It can be demonstrated that: $\frac{1}{A^2_{IIP3,tot}} \overset{*}{\cong} \frac{1}{A^2_{IIP3,1}} + \frac{\alpha_1^2}{A^2_{IIP3,2}}$

non-linearity of latter stages dominates

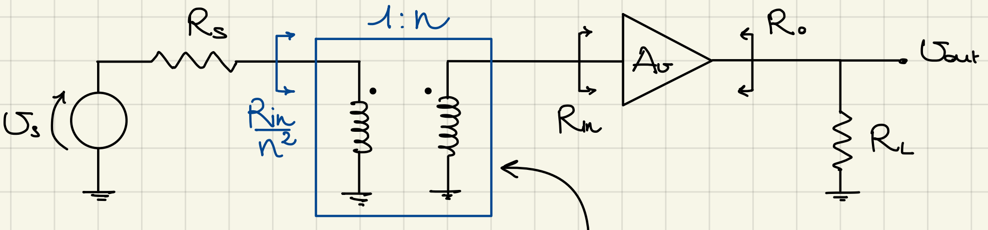
* under hp. of band-pass filtering between the two blocks

↓

otherwise, cascaded 2nd order non-linearities produce the same effect of a 3rd order non-linearity (IM3):

$$\alpha_2 x^2 \rightarrow \omega_2 - \omega_1 \qquad \beta_2 y^2 \rightarrow 2\omega_2 - \omega_1 \Longrightarrow \text{IM3 term}$$

Effects of impedance matching



impedance transformation network (model)

$$\frac{V_{out}}{V_s} = \frac{R_{in}/n^2}{R_{in}/n^2 + R_s} \cdot n \cdot A_v \cdot \frac{R_L}{R_L + R_o}$$

$$= \underbrace{\frac{n R_{in}}{R_{in} + n^2 R_s}}_{\alpha} \cdot A_v \cdot \frac{R_L}{R_L + R_o}$$

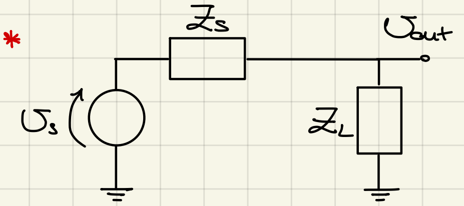
α (input voltage division)

A_v: input-to-output gain

A_o: source-to-output gain

$A_o = \alpha A_v$

*

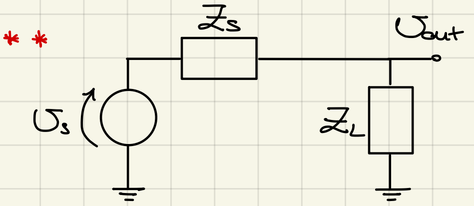


To maximize gain: $|Z_L| \gg |Z_s|$

$R_L \gg R_s$

$$\left.\frac{V_{out}}{V_s}\right|_{max} \longrightarrow 1$$

**



(conjugate matching)

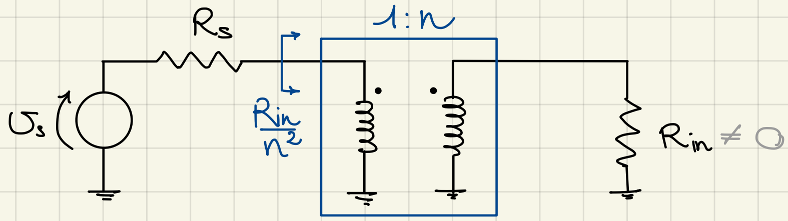
To maximize power transfer: $Z_L = Z_s^*$

$R_L = R_s$

$$\left.P_L\right|_{max} = \frac{|V_{out}|^2}{2R_L} = \frac{|V_s|^2}{8R_L}$$

$$\frac{V_{out}}{V_s} \longrightarrow \frac{1}{2}$$

Impedance matching basically allows us to maximize the power transfer while achieving a better gain, that is closer to the maximum obtainable.



$$\alpha = \frac{n R_{in}}{R_{in} + n^2 R_s}$$

* Try to maximize α (i.e. maximize gain):

$$\frac{\partial \alpha}{\partial n} = \frac{R_{in}(R_{in} + n^2 R_s) - nR_{in} \cdot 2nR_s}{(R_{in} + n^2 R_s)} = 0 \longrightarrow R_{in}^2 = n^2 \cancel{R_{in}} R_s$$

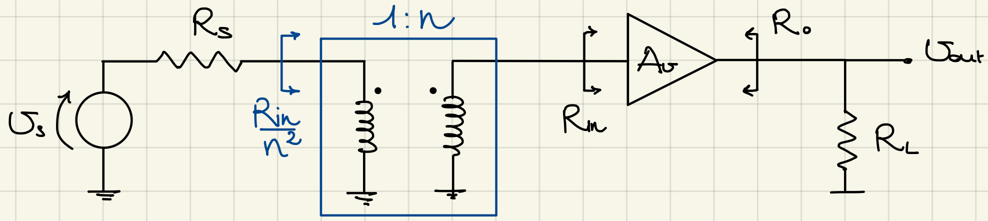
$$\downarrow$$

$$n_{opt} = \sqrt{\frac{R_{in}}{R_s}}$$

$$\Longrightarrow \alpha_{max} = \alpha(n_{opt}) = \frac{1}{2} \cdot \sqrt{\frac{R_{in}}{R_s}} > \frac{1}{2} \text{ if } R_{in} > R_s!$$

Note that $\frac{R_{in}}{n_{opt}^2} = R_s \Longrightarrow$ impedance has been matched **

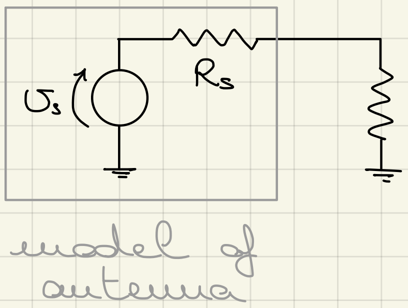
While maximizing the gain, also the power transfer has been maximized!



$$\left.\frac{V_{out}}{V_s}\right|_{max} = \overset{\alpha}{\left(\frac{1}{2} n_{opt}\right)} A_v \frac{R_L}{R_o + R_L}$$

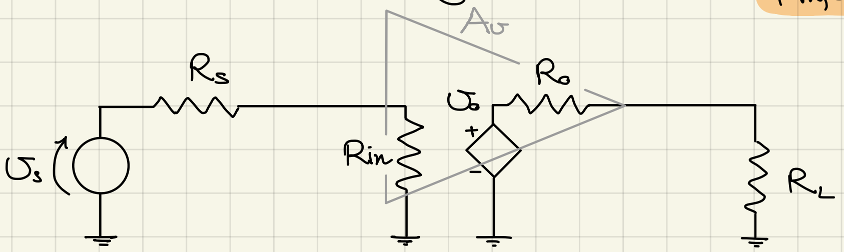
By matching the input resistance, we granted maximum power transfer while increasing the gain by a factor n_{opt}.

Power gain



Available power: $P_{s,av} = \frac{|V_s|^2}{8R_s} = P_{L,max}$

Available power gain: $G_A = \frac{P_{out,av}}{P_{in,av}}$



$\frac{V_o}{V_s} = \alpha A_v = A_o$

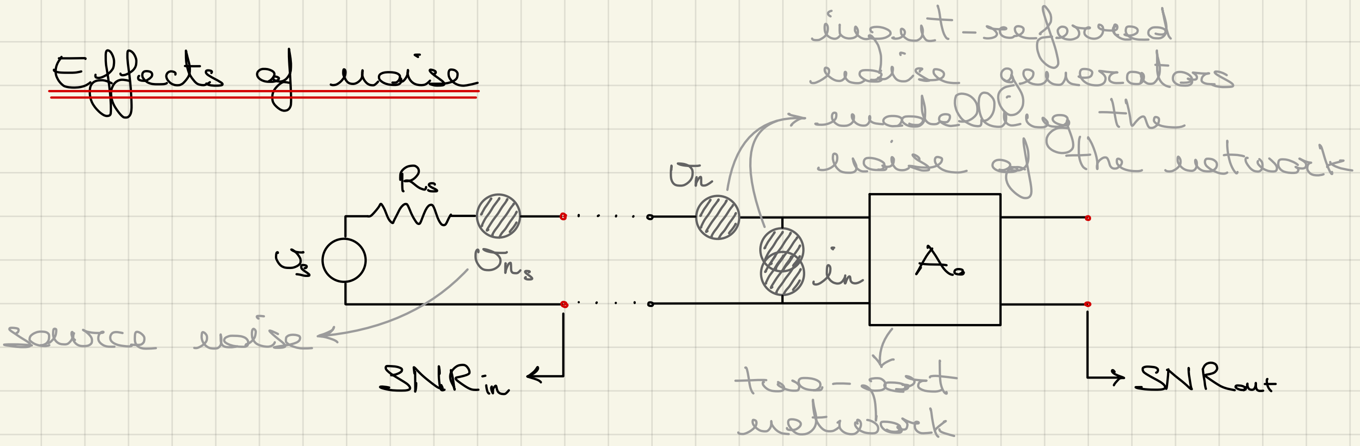
$P_{in,av} = \frac{|V_s|^2}{8R_s}$ $\quad P_{out,av} = \frac{|V_o|^2}{8R_o}$

$$\left[G_A = A_o^2 \frac{R_s}{R_o}\right]$$

In general: av. power gain ≠ square of voltage gain

They are equal only when $R_s = R_o$.

Effects of noise



Noise Figure: $\left[NF := \frac{SNR_{in}}{SNR_{out}}\right] = \frac{v_{s\,in}^2}{\overline{v^2_{n\,in}}} \cdot \frac{\overline{v^2_{n\,out}}}{v^2_{s\,out}} = \frac{1}{A_o^2} \frac{(\overline{v^2_{n\,network}} + \overline{v^2_{n_s}})A_o^2}{\overline{v^2_{n_s}}}$

$$\boxed{NF = 1 + \frac{\overline{v^2_{n\,network}}}{\overline{v^2_{n_s}}}}$$ where $\overline{v^2_{n\,network}} = \overline{(v_n + i_n R_s)^2}$

It is a measure of how much noise the network is adding to the source noise.

Also note that: $\dfrac{\overline{v_{n_{out}}^2}}{A_o^2} = NF \cdot \overline{v_{n_s}^2}$

total noise at the output

total noise at the input

If the source noise is due to just the source resistance:

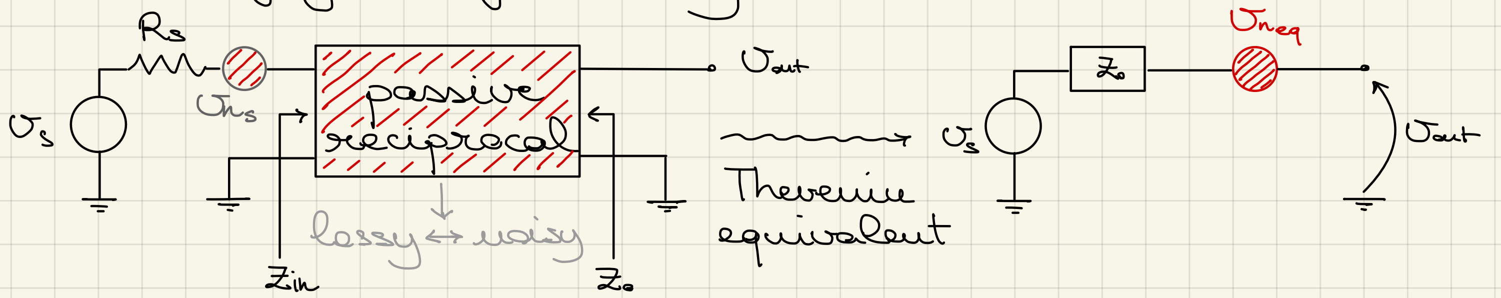
$$PSD_{in} = 4kTR_s \cdot NF$$

total noise density at the input

$\overline{v_n^2}$: noise power

$\dfrac{\overline{v_n^2}}{\Delta f}$: noise PSD

Noise figure of a lossy circuit



Nyquist theorem: $\left[\dfrac{\overline{v_{n_{eq}}^2}}{\Delta f} = 4kT\, Re[Z_o]\right]$

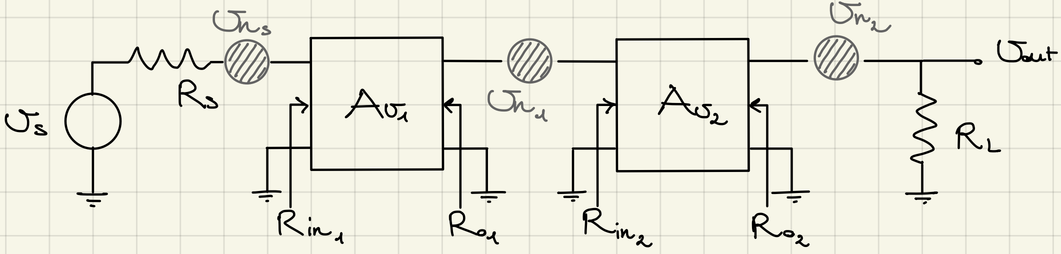
$$NF = \frac{\overline{v_{n_{out}}^2}/A_o^2}{\overline{v_{n_s}^2}} = \frac{\overline{v_{n_{eq}}^2}}{A_o^2}\cdot\frac{1}{\overline{v_{n_s}^2}} = \frac{4kTR_o}{A_o^2}\cdot\frac{1}{4kTR_s} = \frac{1}{\underbrace{A_o^2\frac{R_s}{R_o}}_{G_A}}$$

$\Longrightarrow \left[NF = \dfrac{1}{G_A} = L_A\right]$ available power loss

The noise figure of a lossy circuit is given by its available power loss (= inverse of its available power gain).

e.g.: filter with 2dB power loss → NF = 2dB

Noise figure of cascaded stages



$$NF = \underbrace{1 + \frac{\overline{v_{n_1}^2}/\Delta f}{\underbrace{\alpha_1^2 A_{v_1}^2}_{A_{o_1}}}\cdot\frac{1}{4kTR_s}}_{NF_1} + \frac{\overline{v_{n_2}^2}/\Delta f}{\alpha_1^2 A_{v_1}^2\underbrace{\alpha_2^2 A_{v_2}^2}_{A_{o_2}}}\cdot\frac{1}{4kTR_s}$$

$\alpha_1 = \dfrac{R_{in_1}}{R_{in_1} + R_s}$

$\alpha_2 = \dfrac{R_{in_2}}{R_{in_2} + R_{o_1}}$

$$NF_2 = 1 + \frac{\overline{v_{n2}^2}/\Delta f}{\alpha_2^2 A_{v2}^2} \cdot \frac{1}{4kTR_{o1}} \implies NF = NF_1 + \frac{(NF_2 - 1)}{\alpha_1^2 A_{v1}^2} \cdot \frac{R_{o1}}{R_s}$$

→ G_{A1}

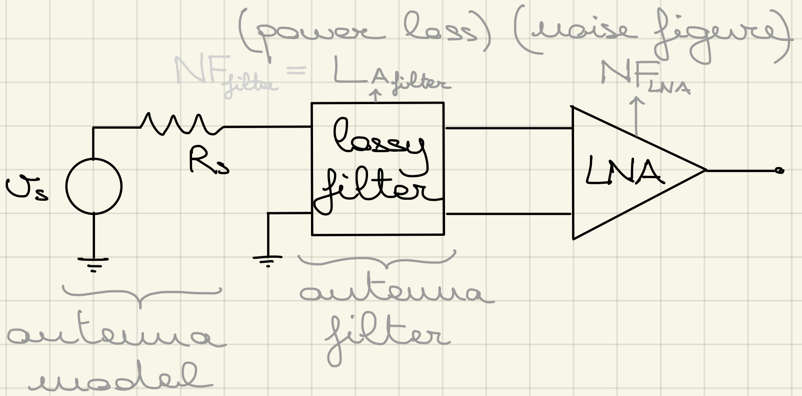
$$\implies \boxed{NF = NF_1 + (NF_2 - 1) L_{A1}}$$

In general, for N cascaded stages:

$$\boxed{NF = 1 + \sum_{i=1}^{N} \left\{ \frac{NF_i - 1}{\prod_{j=1}^{i-1} G_{Aj}} \right\}}$$

→ noise figure of first stages dominates

Example: filter + LNA cascade



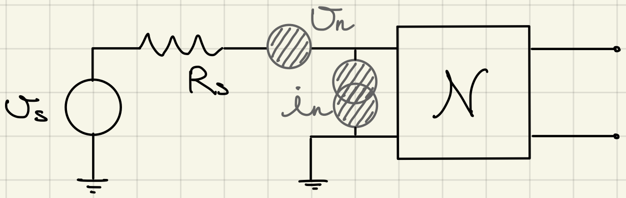
Total noise figure:

$$\boxed{\begin{aligned} NF &= NF_{filter} + \frac{NF_{LNA} - 1}{1/L_{A filter}} \\ &= L_{A filter} + L_{A filter}(NF_{LNA} - 1) \\ &= L_{A filter} \cdot NF_{LNA} \end{aligned}}$$

$$NF_{dB} = L_{A filter\, dB} + NF_{LNA\, dB}$$

⟹ The noise figure of the LNA is amplified by the losses of the previous passive filter (therefore cascaded filters will degrade the total NF even further)

Noise matching



v_n and i_n are correlated (origin from the same physical noise source inside network N)

$$NF = 1 + \frac{\text{network noise}}{\text{source noise}} = 1 + \frac{\overline{(v_n + i_n \cdot R_s)^2}/\Delta f}{4kTR_s}$$

referred e.g. to the input of N

$$\approx 1 + \frac{\overline{v_n^2}/\Delta f + \overline{i_n^2} \cdot R_s^2/\Delta f}{4kTR_s} = 1 + \frac{\overline{v_n^2}/\Delta f}{4kTR_s} + \frac{\overline{i_n^2}/\Delta f}{\frac{4kT}{R_s}}$$

if we assume noise to be uncorrelated.

$$\Longrightarrow NF \approx 1 + \frac{\text{network voltage noise}}{\text{source voltage noise}} + \frac{\text{network current noise}}{\text{source current noise}}$$

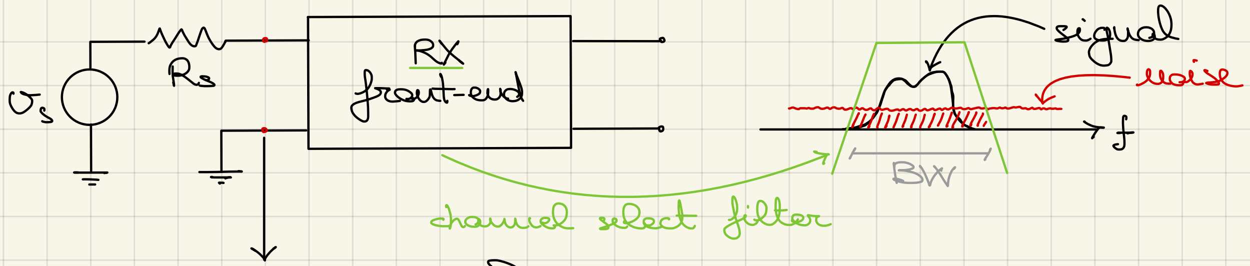
NF has a term decreasing with R_s and a term increasing with R_s.

Therefore, a minimum NF exists for an optimal R_s

$$\frac{\partial NF}{\partial R_s} = 0 \longrightarrow \boxed{R_{s_{opt}} = \sqrt{\frac{\overline{v_n^2}}{\overline{i_n^2}}}}$$

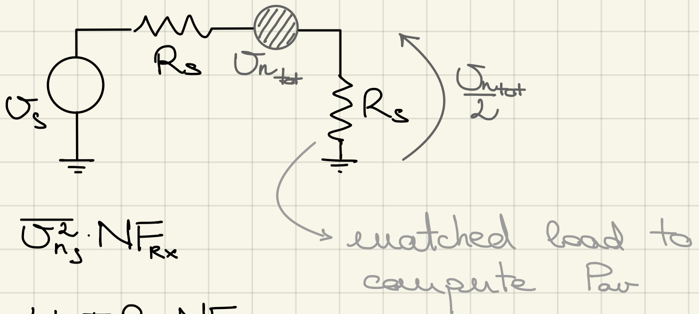
RX sensitivity and Dynamic Range

RX sensitivity = min. detectable signal (SNR_{min})



$$SNR_{min} = \frac{P_{s,av_{min}}}{P_{n,av}}$$

Available noise power:



$$\longrightarrow P_{n,av} = \frac{\overline{v^2_{n_{tot}}}}{R_s} \cdot \frac{1}{4}$$

Remembering that $\overline{v^2_{n_{tot}}} = \overline{v^2_{n_s}} \cdot NF_{RX}$

$$\frac{\overline{v^2_{n_{tot}}}}{\Delta f} = 4KTR_s \cdot NF_{RX}$$

we obtain $\frac{P_{n,av}}{\Delta f} = \frac{\cancel{4}KT\cancel{R_s} \cdot NF_{RX}}{\cancel{4}\cancel{R_s}} = KT \cdot NF_{RX}$

KT ⟶ available power density of the source noise

$$\Longrightarrow P_{n,av} = KT \cdot NF_{RX} \cdot BW$$

$$\longrightarrow SNR_{min} = \frac{P_{s,av_{min}}}{KT \cdot NF_{RX} \cdot BW}$$

$$\longrightarrow \boxed{P_{s,av_{min}} = SNR_{min} \cdot KT \cdot NF_{RX} \cdot BW}$$

$KT = 4 \cdot 10^{-21}\,J$ at 25°C $\longrightarrow 10 \log_{10} KT = -204\ dBw/Hz$

$= -174\ dBm/Hz$

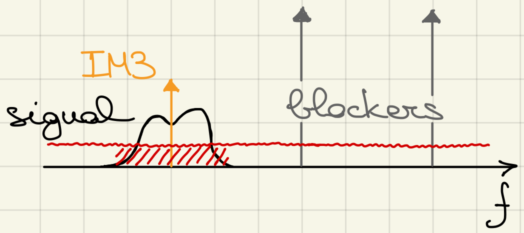
$$\left[P_{s,av_{min}}\Big|_{dBm} = -174\,\frac{dBm}{Hz} + NF_{RX}\Big|_{dB} + SNR_{min}\Big|_{dB} + 10\log_{10} BW \right]$$

Example: GSM handset

- sensitivity $P_s = -100 dBm$
- $BW = 200kHz$
- $SNR_{min} = 9dB$

$$\left.\right\} NF_{RX_{dB}} < P_s + 174 - SNR_{min} - 10\log BW$$
$$< 12dB$$

Dynamic range or SFDR (Spurious-Free Dynamic Range)



$$\left[SFDR_{dB} := P_{in\,max}\Big|_{dB} - P_{in\,min}\Big|_{dB} \right]$$

$P_{in\,max}$: input power of the two tones such that IM3 power equals noise power

$P_{in\,min}$: sensitivity level

Remembering that $P_{IIP3} = P_{in} + \frac{\Delta P}{2}$ (in dB)

$$= P_{in} + \frac{P_{out} - P_{out,IM3}}{2}$$

$$= P_{in} + \frac{P_{in} + G_A - (P_{in,IM3} + G_A)}{2}$$

$$= \frac{3}{2} P_{in} - \frac{1}{2} P_{in,IM3}$$

$P_{in,IM3}$ → input-referred level of IM3 products

At $P_{in} = P_{in_{max}}$ by definition $P_{in,IM3} = P_n$.

$$P_{IIP3} = \frac{3}{2} P_{in\,max} - \frac{1}{2} P_n$$

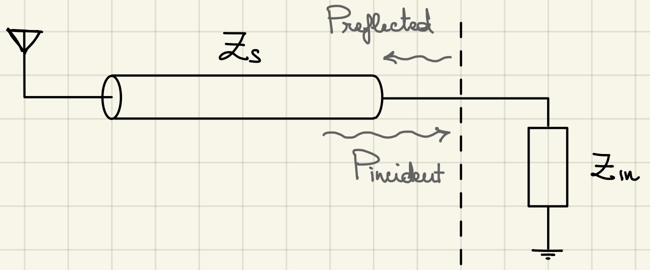
P_n → input-referred level of noise

$$\longrightarrow P_{in\,max} = \frac{1}{3}\left(2 P_{IIP3} + P_n\right)$$

$$SFDR = P_{in\,max} - P_{in\,min} = \frac{2}{3} P_{IIP3} + \frac{1}{3} P_n - \left(P_n + SNR_{min}\right)$$

$$\left[SFDR_{dB} = \frac{2}{3}\left(P_{IIP3}\Big|_{dBm} - P_n\Big|_{dBm} \right) - SNR_{min}\Big|_{dB} \right]$$

Scattering parameters (S-parameters)

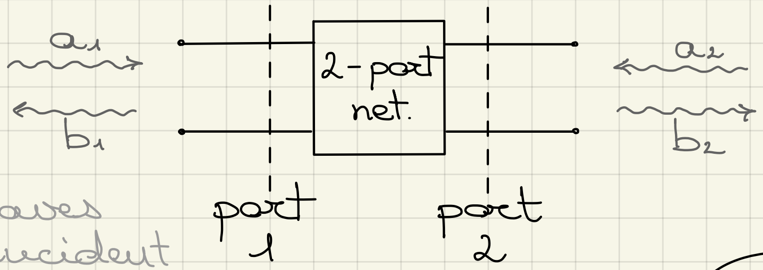


Reflection coefficient

$$\Gamma := \frac{P_{reflected}}{P_{incident}}$$

$$\Gamma = \left|\frac{Z_{in} - Z_s}{Z_{in} + Z_s}\right|^2 \longrightarrow$$ only if $Z_{in} = Z_s$: $\Gamma = 0$ no reflection ("termination" is matched to the characteristic impedance of the line)

Extension to 2-port networks

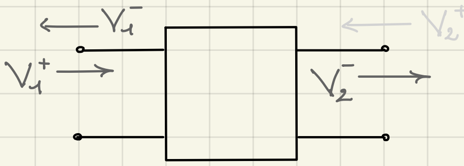


power waves

reflected – incident

$$\begin{cases} b_1 = S_{11} \cdot a_1 + S_{12}\, a_2 \\ b_2 = S_{21} \cdot a_1 + S_{22} \cdot a_2 \end{cases} \longleftrightarrow \underline{\underline{S}} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

→ **scattering parameters**



power coeff. voltage coeff $S = s^2$

$$\sqrt{S_{11}} = \frac{V_1^-}{V_1^+}\Big|_{V_2^+ = 0} =: s_{11}$$

"S_{11} is the reflection coefficient at port 1 with matched port 2"

Same goes for other ones.

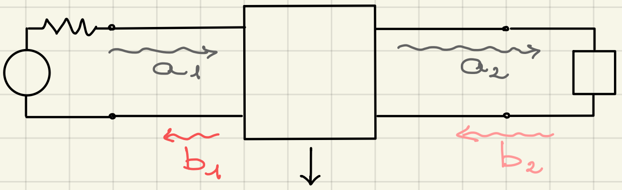
Input return loss: $RL_{in} = 10 \log_{10} \frac{1}{|s_{11}^2|} = -20 \log_{10}|s_{11}|$

Output return loss: $RL_{out} = -20 \log_{10} |s_{22}|$

Forward gain: $20 \log_{10} |s_{21}|$

Reverse isolation: $-20 \log |s_{12}|$

Note: a non-matched load at the output of the network might cause stability issues if the reverse isolation is non-infinite ($s_{12} \neq 0$)



non-matched → load ⇒ positive feed-back loop

⇓

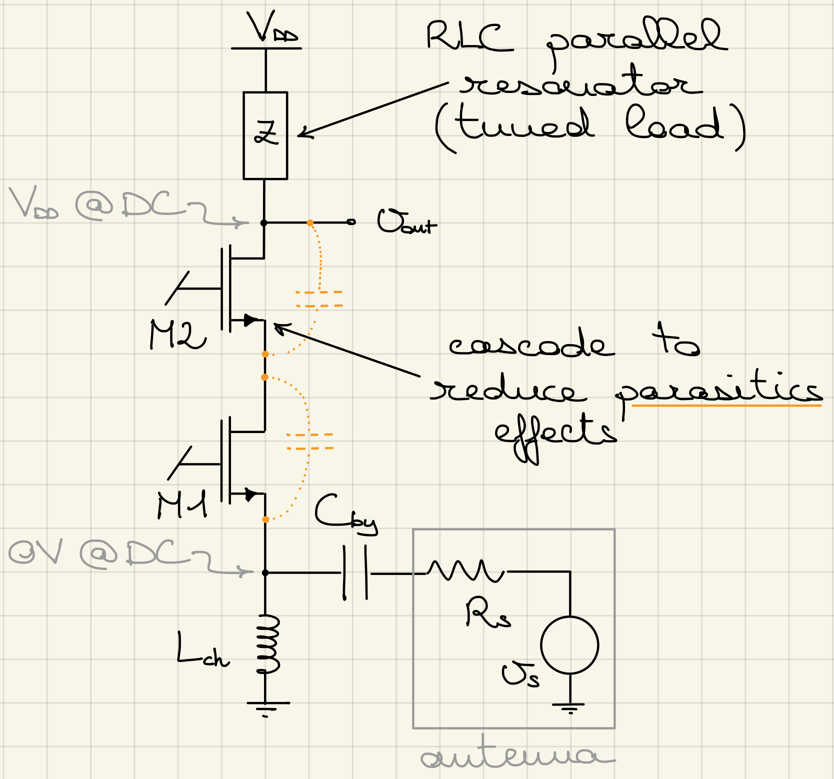
stability issues

↓ non-null reverse gain

Low Noise Amplifiers (LNAs)

Requirements:
- Low noise (NF)
- Large gain (G_A or S_{21})
- Input matching ($1/S_{11}$)
- Linearity (IIP3) because of blockers

Common-gate topology

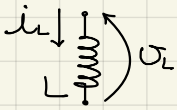


RLC parallel resonator (tuned load)

cascode to reduce parasitics effects

L_{ch}: choke inductor

$$\frac{di_L}{dt} = \frac{v_L}{L}$$



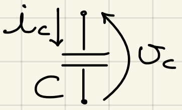
$$L_{ch} \to \infty \Rightarrow \frac{di_L}{dt} \to 0$$

$$\Rightarrow i_L \to \text{const.}$$

Sufficiently large inductor is treated as a current generator (open circuit in AC)

C_{by}: bypass capacitor

$$\frac{dv_C}{dt} = \frac{i_C}{C}$$

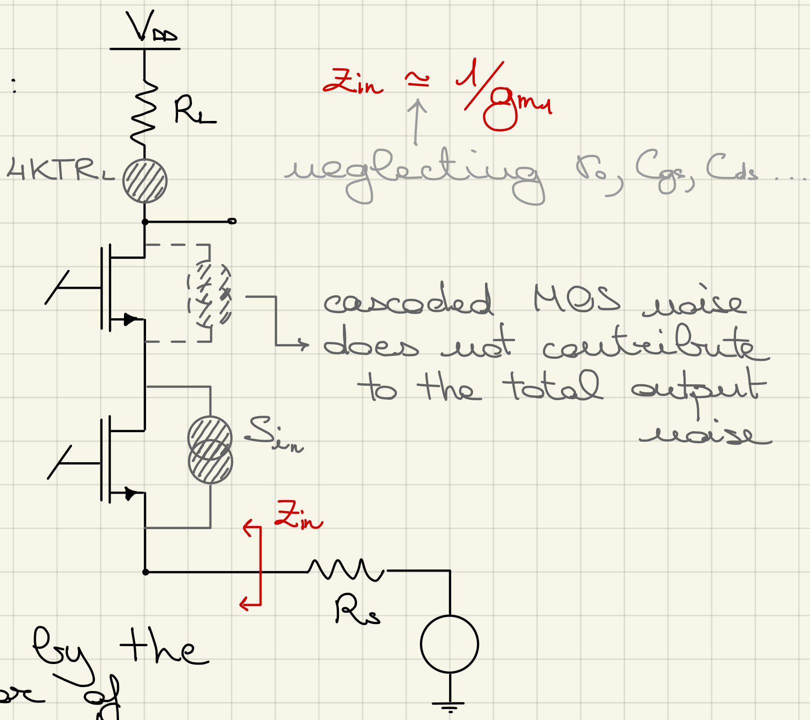


$$C_{by} \to \infty \Rightarrow \frac{dv_C}{dt} \to 0$$

$$\Rightarrow v_C \to \text{const.}$$

Sufficiently large capacitor is treated as a voltage generator (short circuit in AC)

At center frequency ω_0:



- Matching condition:

$$1/g_{m1} = R_s$$

Voltage gain:

$$A_0 = \frac{v_{out}}{v_s} \underset{\text{matched input}}{\overset{\uparrow}{=}} \frac{R_L}{2R_s}$$

limited by the Q factor of the resonator
$\left(Q = \omega_0 R_L C = \frac{R_L}{\omega_0 L} \lesssim 10\right)$

$S_{i_n} = 4kT\gamma g_{d0}$ ("van der Ziel" MOSFET noise model)

where $g_{d0} = \left.\frac{\partial I_D}{\partial V_{DS}}\right|_{V_{DS}=0}$ valid for any operating region

- Triode: $I_D = K\left[2V_{ov}V_{DS} - V_{DS}^2\right] \longrightarrow g_{d0} = 2KV_{ov} = \frac{1}{r_{on}}$
  - $\hookrightarrow \gamma = 1$
- Saturation: $I_D = KV_{ov}^2 \longrightarrow g_{d0} = g_m$
  - $\hookrightarrow \gamma = \frac{2}{3}$

In case of carrier velocity saturation:

$$g_{d0} = \frac{g_m}{\alpha} \geq g_m \Longrightarrow S_{i_n} = 4kT\frac{\gamma}{\alpha}g_m$$

- Triode → $\alpha = 1$
- Saturation → $\alpha < 1$

Noise figure:

$$NF = 1 + \frac{4kT\,\gamma/\alpha\;1/g_{m1}}{4kTR_s} + \frac{4kTR_L/A_0^2}{4kTR_s}$$

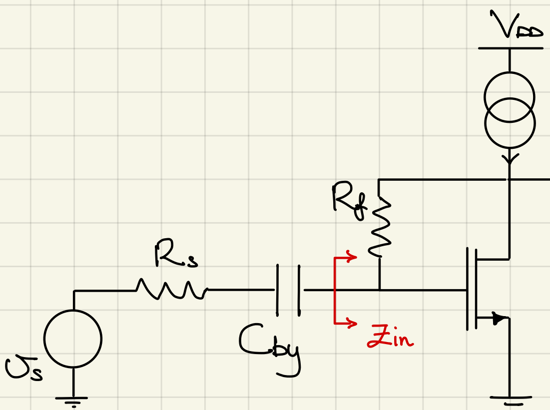
(referred to the input)

$$\underset{\text{matched input}\rightarrow}{=} 1 + \frac{\gamma}{\alpha} + 4\frac{R_s}{R_L}$$

term enforced by necessity of impedance matching

term inversely proportional to A_0 hence limited by the Q factor

Shunt feedback topology

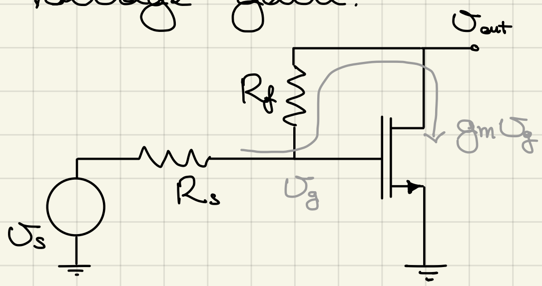


$Z_{in} = 1/g_m$

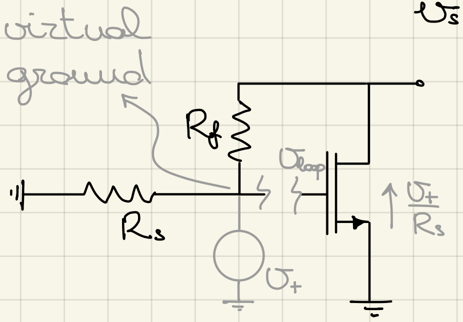
- Matching condition:

$$1/g_m = R_s$$

- Voltage gain:



$$A_0 = \frac{v_{out}}{v_s} = \frac{1 - g_m R_f}{1 + g_m R_s} \Longleftarrow \begin{cases} v_{out} = v_s - g_m v_g (R_s + R_f) & \text{KCL} \\ \dfrac{v_s - v_g}{R_s} = \dfrac{v_g - v_{out}}{R_f} & \text{KVL} \end{cases}$$



$$= \frac{G_{direct}}{1 - G_{loop}} + \frac{G_{id}}{1 - \frac{1}{G_{loop}}}$$

turn MOS off

$$G_{loop} = -\frac{1}{g_m R_s} \qquad G_{id} = -\frac{R_f}{R_s} \qquad G_{direct} = 1$$

With matched input: $G_{loop} = -1 \Longrightarrow A_0 = \frac{1}{2}\left(1 - \frac{R_f}{R_s}\right)$

(for $R_f \gg R_s$, $A_0 \simeq -\frac{1}{2}\frac{R_f}{R_s} < 0$ → inverting stage)

- Noise figure:

$$NF = 1 + \frac{4KT\,\gamma/\alpha\, g_m \cdot \left(\frac{R_f + R_s}{1 - G_{loop}}\right)^2}{4KT R_s A_0^2} + \frac{4KT R_f}{4KT R_s A_0^2}$$

(referred to the output)

matched input + $R_f \gg R_s$ →

$$= 1 + \frac{\gamma}{\alpha} + 4\frac{R_s}{R_f}$$

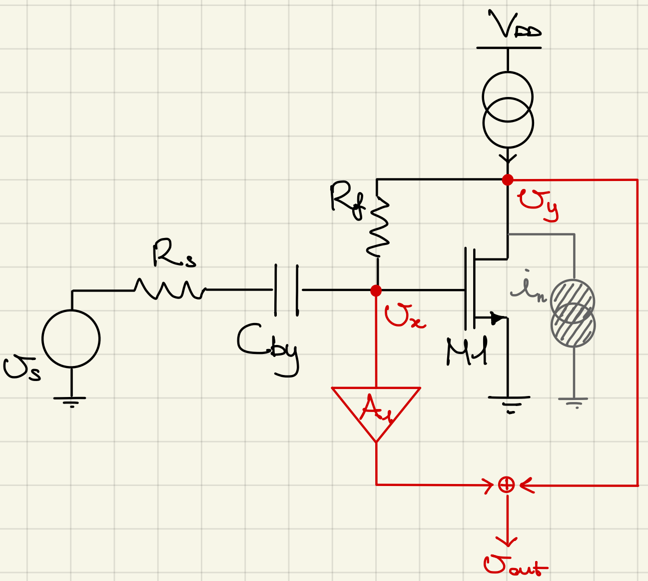
(same as Common Gate, same limitations)

To overcome NF limits:

- noise cancelling
- impedance transformation
- feedback (to decouple $1/g_m$ from R_s)

Noise cancelling

Take e.g. Shunt Feedback configuration:



find 2 nodes to be combined such that noise source (i_n) is cancelled, but signal (v_s) is not cancelled

Noise transfers

$$\frac{v_y}{i_n} = \frac{R_s+R_f}{1-G_{loop}} > 0$$

$$\frac{v_x}{i_n} = \frac{v_y}{i_n} \cdot \frac{R_s}{R_s+R_f} = \frac{R_s}{1-G_{loop}} > 0$$

$$\boxed{A_1 = -\left(1+\frac{R_f}{R_s}\right)} \iff v_{out}\Big|_{i_n} = 0 \longleftarrow \frac{v_{out}}{i_n} = A_1 \cdot \frac{v_x}{i_n} + \frac{v_y}{i_n} = A_1 \frac{R_s}{1-G_{loop}} + \frac{R_s+R_f}{1-G_{loop}}$$

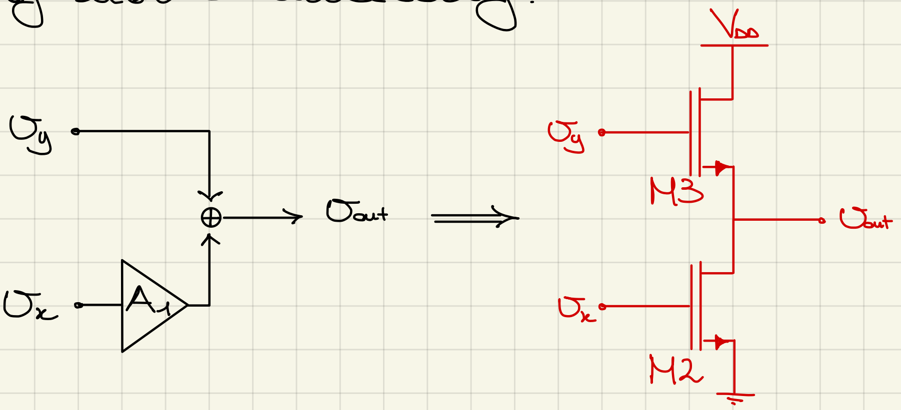
Signal transfers

$$\frac{v_{out}}{v_s} = A_1 \frac{v_x}{v_s} + \frac{v_y}{v_s} = -\left(1+\frac{R_f}{R_s}\right) \cdot \frac{1/g_{m1}}{1/g_{m1}+R_s} + A_0$$

new voltage gain

$$\text{matched input} \rightarrow = -\left(\cancel{1}+\frac{R_f}{R_s}\right)\frac{1}{2} + \frac{1}{2}\left(\cancel{1}-\frac{R_f}{R_s}\right) = -\frac{R_f}{R_s} = A_0' \simeq 2A_0!$$

How can we implement sum and multiplication, without adding extra noise that would spoil the concept of noise cancelling?

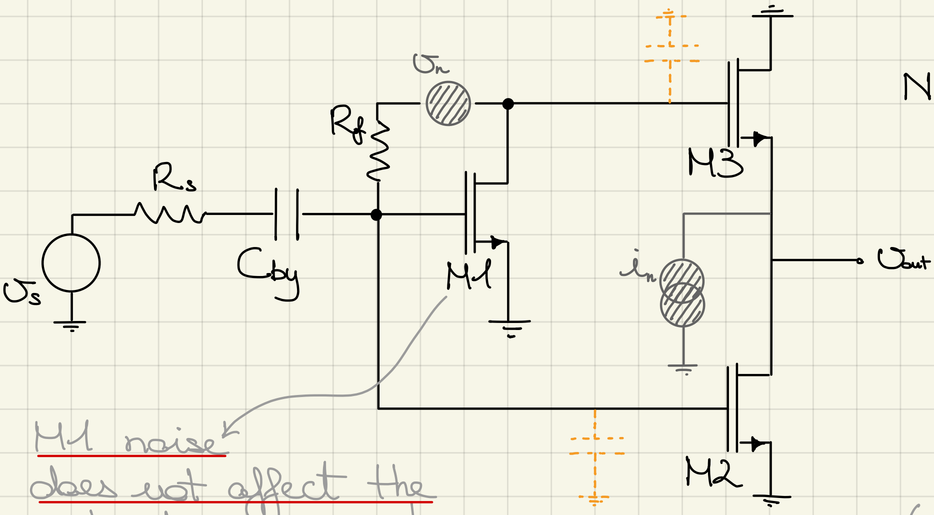


By applying superposition principle: $\frac{v_{out}}{v_y} \simeq 1 \quad (r_{o_2} \to \infty)$

$$\frac{v_{out}}{v_x} = -\frac{g_{m2}}{g_{m3}} = A_1$$

$$\Longrightarrow v_{out} = v_y - \frac{g_{m2}}{g_{m3}} v_x$$

We now need to compute the NF of the LNA with the addition of the noise cancelling circuit.



M1 noise does not affect the output anymore!

$$NF = 1 + \frac{\cancel{4KT}\cancel{R_f}}{\cancel{4KT}R_s\underbrace{\left(\frac{R_f}{R_s}\right)}_{A_0'}^{\cancel{2}}} + \frac{\cancel{4KT}\,\gamma/\alpha(g_{m2}+g_{m3})}{\cancel{4KT}R_s\left(\frac{R_f}{R_s}\right)^2}\,1/g_{m3}^2$$

$$= 1 + \frac{R_s}{R_f} + \overbrace{\left(\frac{g_{m2}}{g_{m3}}+1\right)}^{-A_1}\frac{1}{g_{m3}}\frac{R_s}{R_f^2}\frac{\gamma}{\alpha}$$

$$= 1 + \frac{R_s}{R_f} + \left(2+\frac{R_f}{R_s}\right)\frac{1}{g_{m3}}\frac{R_s}{R_f^2}\frac{\gamma}{\alpha}$$

(referred to the output)

For $R_f \gg R_s$: $NF \simeq 1 + \frac{\cancel{R_f}}{\cancel{R_s}}\frac{1}{g_{m3}}\frac{\cancel{R_s}}{R_f^{\cancel{2}}}\frac{\gamma}{\alpha} = 1 + \frac{\gamma}{\alpha}\cdot\frac{1}{g_{m3}R_f}$

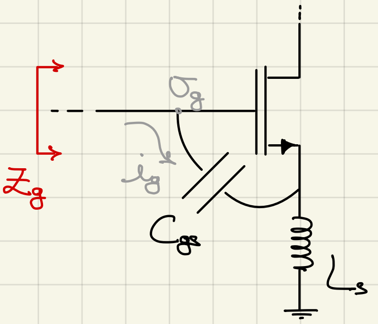
If $g_{m3} > 1/R_f$ then the NF of this stage (independent of $g_{m1} = 1/R_s$!) is lower than the NF of the shunt feedback topology without noise cancelling

Issue: parasitic capacitances

The noise reduction of this technique is limited by parasitics when g_{m3} becomes very large (for a lower NF).

Impedance transformation

Exploit gate-source capacitance of a transistor with inductive degeneration

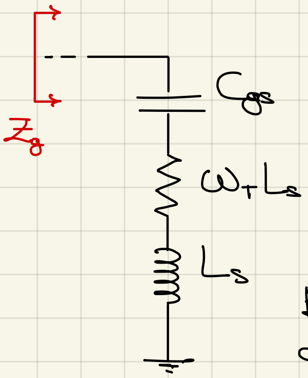


$$\begin{cases} v_g = v_{gs} + sL_s(g_m v_{gs} + i_g) \\ i_g = sC_{gs}\cdot v_{gs} \end{cases}$$

$$\Rightarrow v_g = v_{gs} + (sL_s g_m + s^2 C_{gs} L_s)\, v_{gs}$$

$$Z_g = \frac{v_g}{i_g} = \frac{(1 + sL_s g_m + s^2 C_{gs} L_s)\cancel{v_{gs}}}{sC_{gs}\cdot\cancel{v_{gs}}} = \frac{1}{sC_{gs}} + g_m\frac{L_s}{C_{gs}} + sL_s$$

$\Longrightarrow$ it's a series RLC!

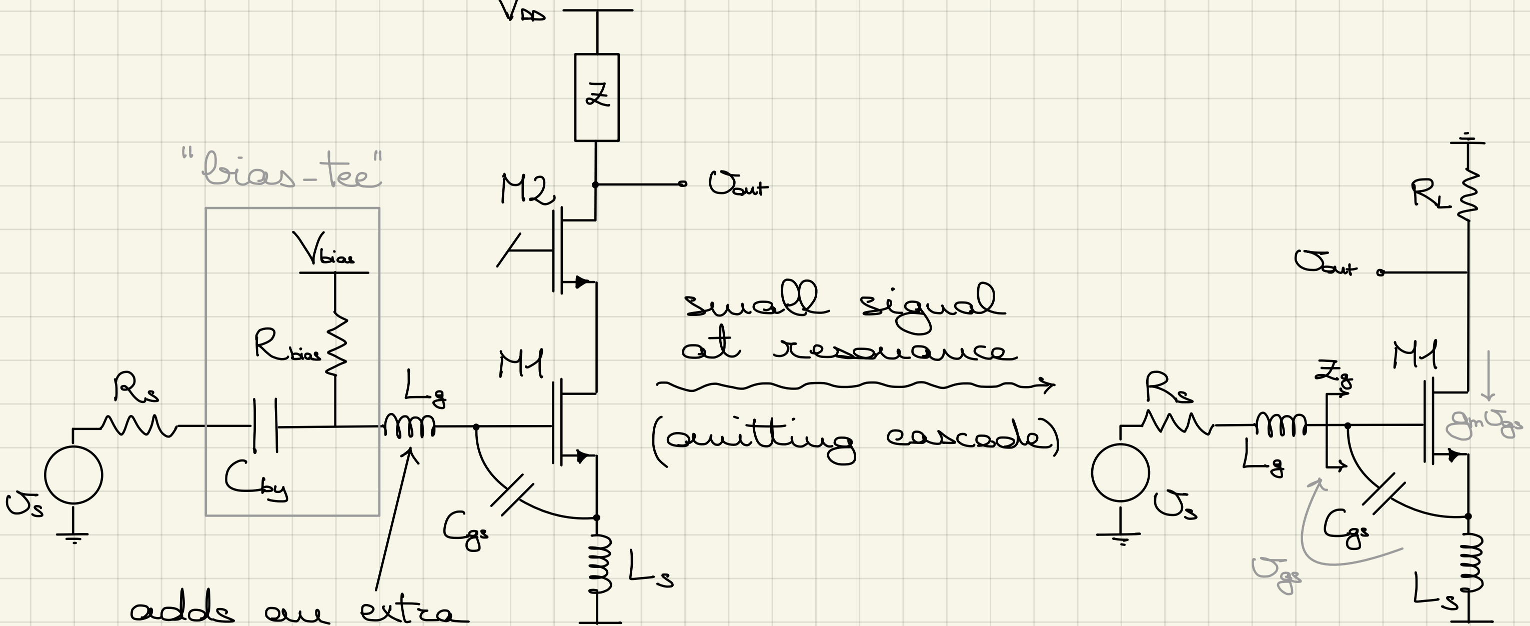


where $\omega_T \simeq \frac{g_m}{C_{gs}}$ (cut-off frequency neglecting C_{gd})

By making L_s and C_{gs} resonate, we can obtain an input impedance that is different from $1/g_m$.

New matching condition:
$$\omega_0 = \frac{1}{\sqrt{L_s C_{gs}}}$$
$$\omega_T L_s = R_s$$

Take e.g. Common Gate configuration:



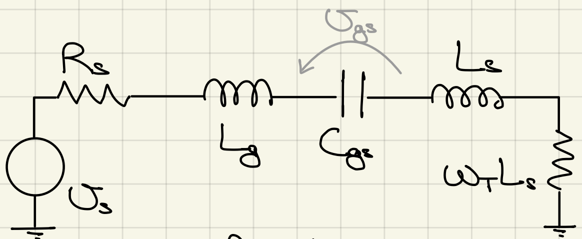
small signal at resonance

(omitting cascode)

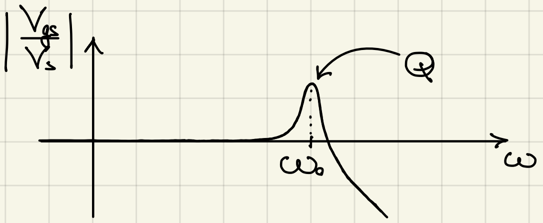
adds an extra degree of freedom when choosing L_s and C_{gs}

- Matching condition:
  - $\omega_0 = \frac{1}{\sqrt{(L_s+L_g)C_{gs}}}$
  - $\omega_T L_s = R_s$

Voltage gain:



equivalent input network

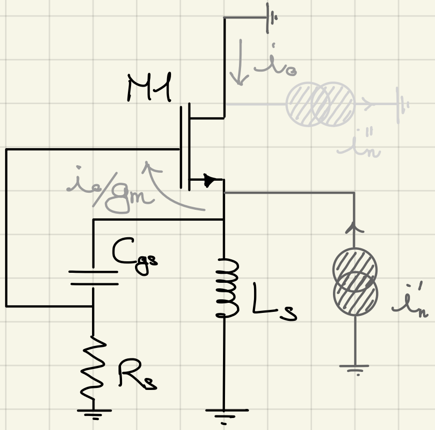


at resonance → $v_{gs} = Q v_s$ where $Q = \frac{1}{\omega_0 C_{gs}(R_s + \omega_T L_s)}$ matched input → $= \frac{1}{\omega_0 C_{gs} \cdot 2R_s}$

$$v_{out} = -g_m v_{gs} R_L = -g_m R_L Q v_s$$

$$\Longrightarrow A_0 = \frac{v_{out}}{v_s} = -g_m R_L Q = -g_m R_L \frac{1}{\omega_0 C_{gs} 2R_s} = -\overbrace{\frac{\omega_T}{\omega_0}}^{\text{increase factor}} \underbrace{\frac{R_L}{2R_s}}_{\text{gain of standard CG topology}}$$

- Noise figure: what we really care about!



(omitting L_g for simplicity)

$$i_o + i_n' = -\frac{i_o}{g_m}\cdot sC_{gs} + \frac{-\frac{i_o}{g_m}\cdot sC_{gs}\cdot R_s - \frac{i_o}{g_m}}{sL_s}$$

$$i_o = -\frac{s\,g_m/C_{gs}}{s^2 + s\left(g_m/C_{gs} + R_s/L_s\right) + \frac{1}{L_sC_{gs}}}\, i_n'$$

$$\left.\frac{i_o}{i_n'}\right|_{\underset{\frac{1}{\sqrt{L_sC_{gs}}}}{\overset{}{\omega=\omega_0}}} = -\frac{\cancel{j\omega_0}\, g_m/C_{gs}}{\cancel{j\omega_0}\left(g_m/C_{gs} + R_s/L_s\right)} = -\frac{\omega_T L_s}{\omega_T L_s + R_s} = -\frac{1}{2}$$

↑ matched input

(Note that i_n' is just half of M1 noise contribution; the other contribution has a transfer to the short circuit output current equal to 1. By summing the two contributions, the overall transfer is still $\frac{1}{2}$)

at resonance

$$\Longrightarrow NF = 1 + \frac{\cancel{4KT}\,\gamma/\alpha\, g_m\left(\frac{1}{2}\right)^2}{\cancel{4KT}\,R_s\left(\frac{g_m}{\omega_0 C_{gs} 2R_s}\right)^2} + \frac{\cancel{4KT}/R_L}{\cancel{4KT}\,R_s\left(\frac{g_m}{\omega_0 C_{gs} 2R_s}\right)^2}$$

(referred to the short circuit output current)

$$= 1 + \frac{\gamma}{\alpha}\cdot\frac{R_s\omega_0^2C_{gs}^2}{g_m} + \frac{4R_s\omega_0^2C_{gs}^2}{R_L g_m^2}$$

$$= 1 + \frac{\gamma}{\alpha}\frac{\omega_0}{\omega_T}\,\omega_0 C_{gs}R_s + \frac{4R_s}{R_L}\left(\frac{\omega_0}{\omega_T}\right)^2$$

reduction factor

quality factor of (matched) entire network

$$= 1 + \underbrace{\frac{\gamma}{\alpha}}_{}\overbrace{\frac{\omega_0}{\omega_T}}^{}\frac{1}{Q_L} + \frac{4R_s}{R_L}\left(\frac{\omega_0}{\omega_T}\right)^2 \quad \text{where} \quad Q_L = \frac{1}{\omega_0 C_{gs} R_s} = 2Q$$

noise term of standard CG topology

quality factor of (matched) Z_g network

For LNAs, we define the transducer power gain as the ratio between output power and available input power:

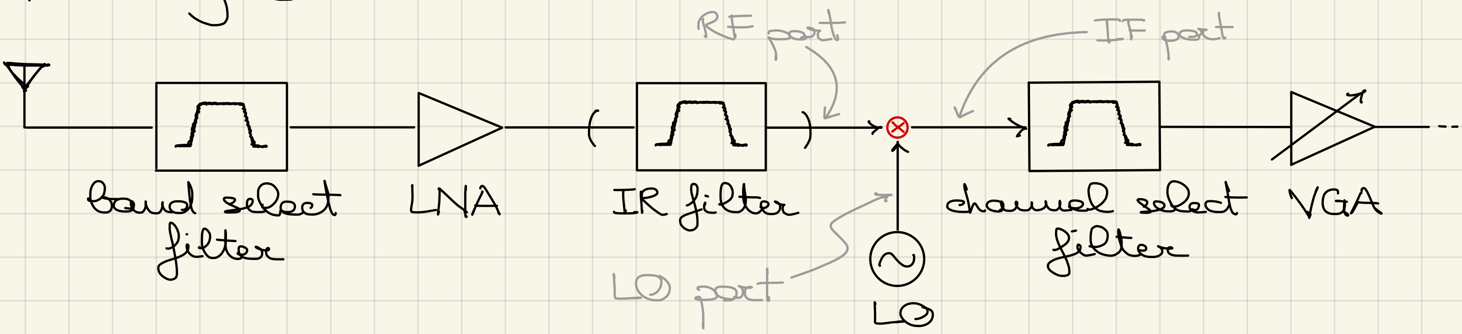
$$G_T := \frac{P_{out}}{P_{in,av}} \leq G_A$$

and the operating power gain as the ratio between output power and input power

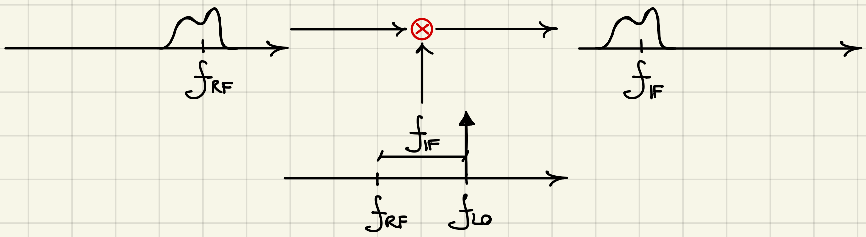
$$G_P := \frac{P_{out}}{P_{in}} \geq G_T$$

Mixers

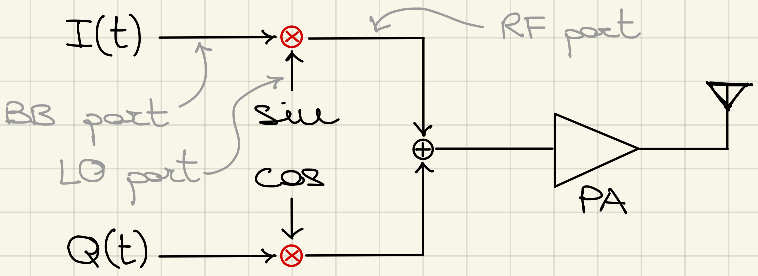
Heterodyne RX structure:



Mixer is used as a DOWN-converter.



Direct-conversion TX structure:



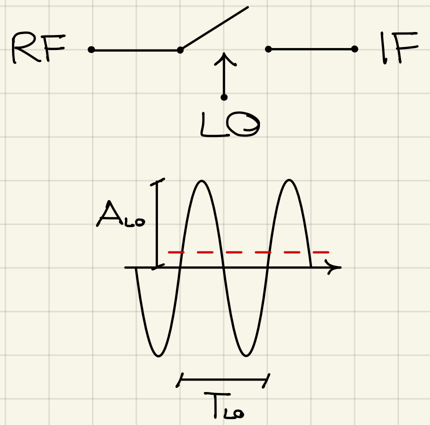
Mixer is used as an UP-converter.

Specifications:

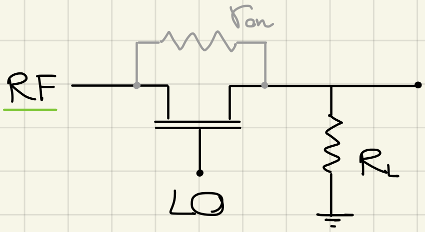
- Conversion gain: $G := \frac{P_{out}}{P_s}$ power at f_{IF} ; power at f_{RF}
- linearity because of blockers (signal at RF port is linearly transferred to IF port)
- Noise figure because LNA gain (in RX) is limited
- Feedthroughs: unwanted signal transfer from one port to another one (signal at RF input leaks into IF output: at IF port there is a signal component at f_{RF}

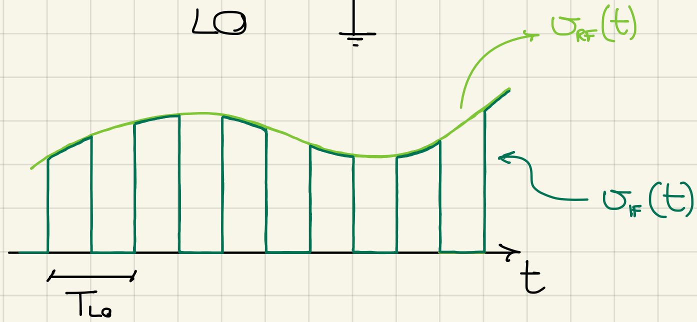
Passive Return-to-Zero (RZ) mixer

(RX case)

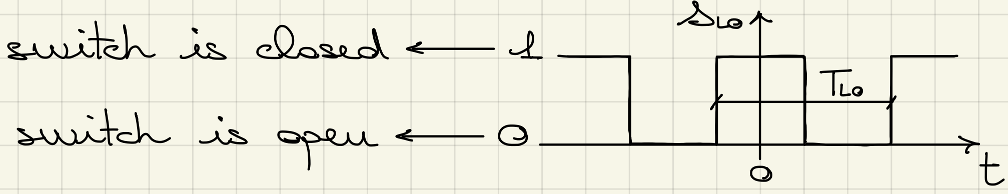


Close the switch when threshold is reached





Defining the function $s_{LO}(t)$:



$$\omega_{LO} = \frac{2\pi}{T_{LO}}$$

(Duty Cycle depends on threshold)

$$\Longrightarrow v_{IF}(t) = \frac{R_L}{R_L + r_{on}} \cdot s_{LO}(t) \cdot v_{RF}(t)$$

$\Longrightarrow$ LTV system (ideally if r_{on} is constant)

$$v_{IF}(t) \simeq \frac{R_L}{R_L + r_{on}} \cdot \left[\frac{1}{2} + \frac{2}{\pi} \cos(\omega_{LO} t) + \text{harmonics} \right] \cdot v_{RF}(t)$$

50% Duty Cycle ($s_{LO}(t)$ is exact square wave function)

RF-to-IF feedthrough

wanted f component

might cause some problems (interferers)

Assume $v_{RF}(t) = A\cos(\omega_{RF} t)$

$$\Longrightarrow v_{IF}(t) = \frac{R_L}{R_L + r_{on}} \cdot \Big[\frac{A}{2} \cos \omega_{RF} t + $$
$$+ \frac{1}{2} A \frac{2}{\pi} \cos(\omega_{LO} - \omega_{RF})t +$$
$$+ \frac{1}{2} A \frac{2}{\pi} \cos(\omega_{LO} + \omega_{RF})t + \text{other terms} \Big]$$

- Conversion (voltage) gain: $A_v = \frac{V_{IF}(\omega_{LO} - \omega_{RF})}{V_{RF}(\omega_{RF})} = \frac{1}{\pi} \frac{R_L}{R_L + r_{on}} < 1$

$$A_{v_{max}} \xrightarrow{r_{on} \to 0} \frac{1}{\pi} \simeq -10\text{ dB}$$

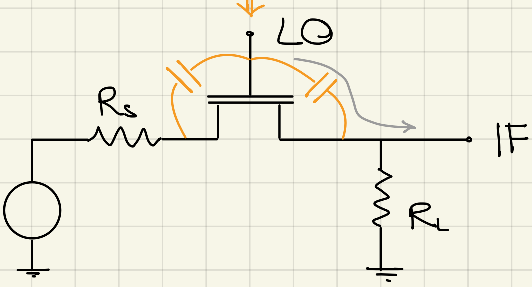
- Linearity: r_{on} depends in reality on v_{gs} i.e. on v_{RF} hence linearity improves for $r_{on} \ll R_L$

$\downarrow$

large parasitic capacitance $\Leftarrow$ large MOSFET

$\downarrow$

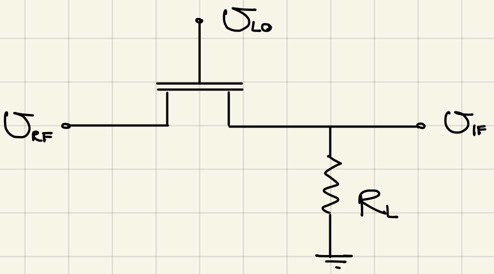
Feedthrough:



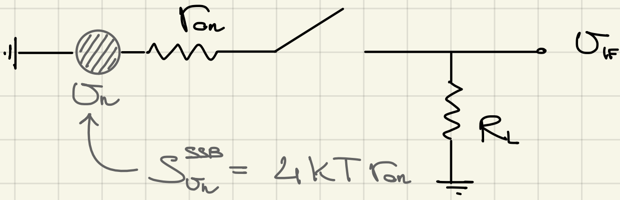
LO - to - IF
LO - to - RF

$\longrightarrow$ Linearity - Feedthrough trade-off

- Noise.



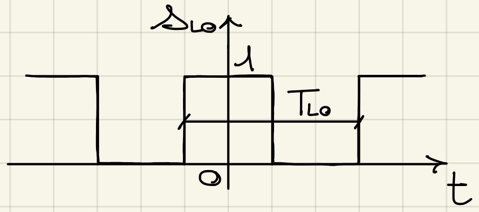
- MOS noise



$$v_{IF_n}\Big|_{MOS} = v_n(t)\cdot s_{LO}(t)\cdot\frac{R_L}{R_L+r_{on}} \Longrightarrow S^{DSB}_{v_{IF}}\Big|_{MOS} = S^{DSB}_{v_n}(f) * |S_{LO}(f)|^2\cdot\left(\frac{R_L}{R_L+r_{on}}\right)^2 =$$

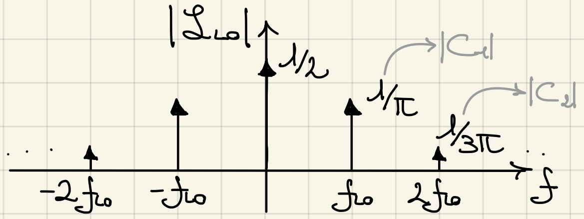
$$= 2kTr_{on} * \sum_{n=-\infty}^{+\infty}|C_n|^2\delta(f-nf_{LO})\cdot\left(\frac{R_L}{R_L+r_{on}}\right)^2$$

$$= 2kTr_{on}\underbrace{\sum_{n=-\infty}^{+\infty}|C_n|^2}_{\text{power of } s_{LO}(t)}\cdot\left(\frac{R_L}{R_L+r_{on}}\right)^2$$



$\downarrow$

$$\int_{-\infty}^{+\infty}|S_{LO}(f)|^2df \underset{\substack{\uparrow \\ \text{Parseval's theorem}}}{=} \frac{1}{T_{LO}}\int_{-\frac{T_{LO}}{2}}^{\frac{T_{LO}}{2}}s^2_{LO}(t)\,dt$$



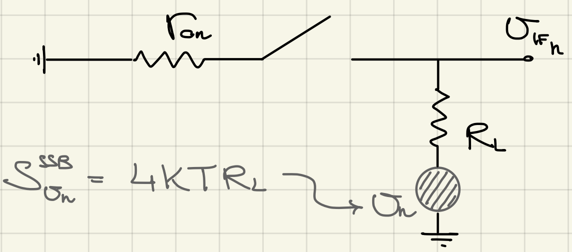
$$\Longrightarrow \sum_{n=-\infty}^{+\infty}|C_n|^2 = \frac{1}{T_{LO}}\cdot\frac{T_{LO}}{2} = \frac{1}{2}$$

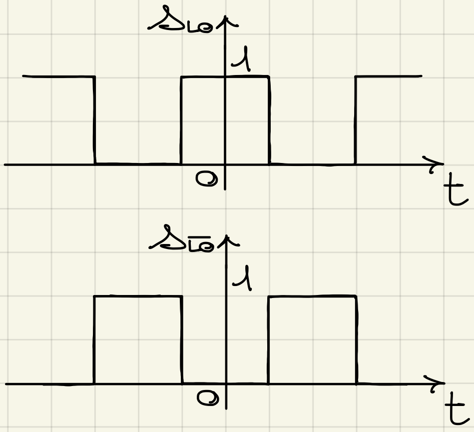
$$\Longrightarrow S^{DSB}_{v_{IF}}(f)\Big|_{MOS} = 2kTr_{on}\cdot\underset{\downarrow}{\frac{1}{2}}\left(\frac{R_L}{R_L+r_{on}}\right)^2$$

v_n is transferred to v_{IF} only for half of the time (50% DC)

Since S_{v_n} is white, the convolution with infinite deltas results in the sum of infinite white noise components that are monotonically decreasing ("spectrum folding")

- R_L noise





$$v_{IF_n}\Big|_{R_L} = \underbrace{v_n(t)\cdot s_{LO}(t)\,\frac{r_{on}}{r_{on}+R_L}}_{\text{when switch is closed}} + \underbrace{v_n(t)\,s_{\overline{LO}}(t)}_{\text{when switch is open}}$$

$$\Longrightarrow S_{v_{IF}}^{DSB}(f)\Big|_{R_L} = 2KTR_L\cdot\frac{1}{\cancel{2}}\cdot\left(\frac{r_{on}}{r_{on}+R_L}\right)^2 + \cancel{2}KTR_L\cdot\frac{1}{\cancel{2}}$$

- Total noise

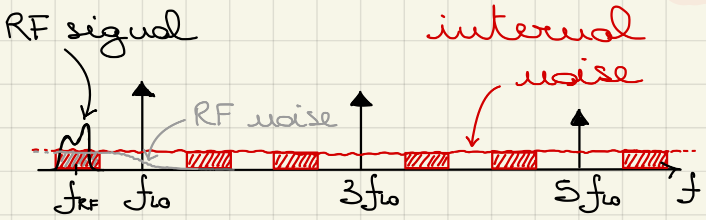
$$\Longrightarrow S_{v_{IF}}^{SSB} = 2KTr_{on}\left(\frac{R_L}{R_L+r_{on}}\right)^2 + 2KTR_L\left(\frac{r_{on}}{r_{on}+R_L}\right)^2 + 2KTR_L$$

$$= \underbrace{2KT\,(r_{on}\,/\!/\,R_L)}_{\text{half PSD when switch is closed}} + \underbrace{2KTR_L}_{\text{half PSD when switch is open}}$$

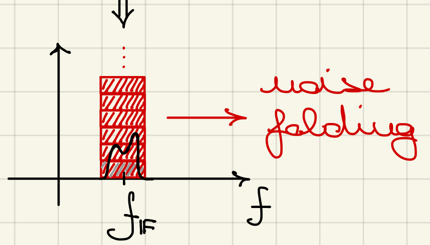
Input referred: $S_{v_{RF}}^{SSB} = \frac{S_{v_{IF}}^{SSB}}{(A_v)^2} = \frac{2KT\,(R_L/\!/r_{on} + R_L)}{\left(\frac{1}{\pi}\frac{R_L}{R_L+r_{on}}\right)^2}$

$$\frac{\partial S_{v_{RF}}^{SSB}}{\partial R_L} = 0 \longrightarrow S_{v_{RF}}^{SSB}\Big|_{min} \simeq \underline{115}\cdot KT\cdot r_{on} \quad \text{for} \quad R_L = \sqrt{2}\,r_{on}$$

mixer noise is typically considerably larger than source noise
⇓
NF contribution is relevant



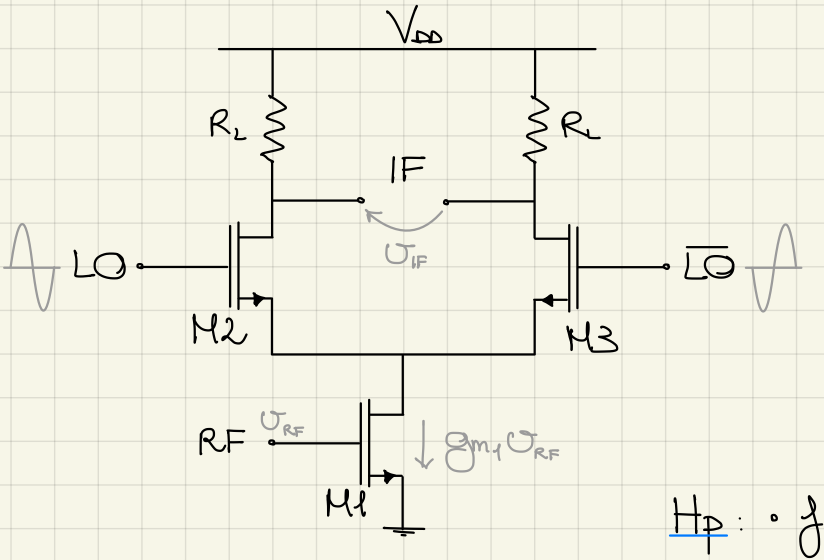
⇓



(internal noise of mixer is downconverted and folded by harmonics; RF noise is not folded by having a LPF at the RF port)

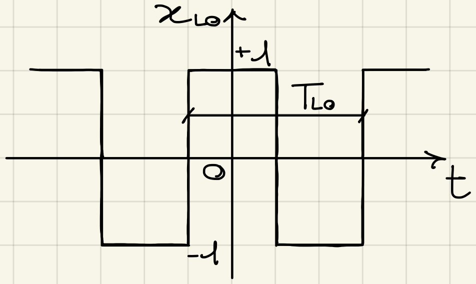
Active mixers: We discriminate between passive and active mixers by their gain (lower and greater than one, respectively) and by the presence of DC bias current in the stage.

Single-Balanced mixers



Active

$$v_{IF}(t) \simeq g_{m1} v_{RF}(t) \cdot x_{LO}(t) \cdot R_L$$



Hp:
- full switching of M2/M3
- 50% Duty Cycle
- M1 is always in saturation

$$v_{RF}(t) = A\cos(\omega_{RF}t)$$

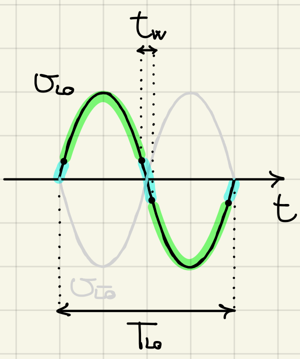
$$v_{IF}(t) = g_{m1}R_L \cdot A\cos\omega_{RF}t \cdot \left[\frac{4}{\pi}\cos\omega_{LO}t - \frac{4}{3\pi}\cos 3\omega_{LO}t + \dots\right]$$

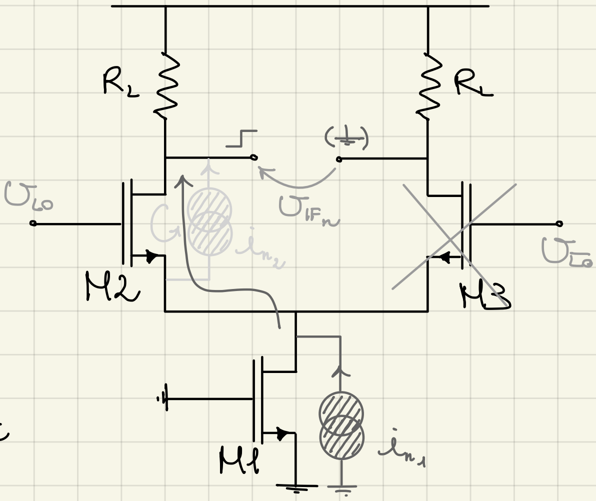
$$= g_{m1}R_L A\,\frac{4}{\pi}\cdot\frac{1}{2}\cos(\omega_{LO}-\omega_{RF})t + \text{other terms}$$

(ideally) no RF-to-IF feedthrough! ⟺ "SINGLE-BALANCED" (LO signal is balanced)

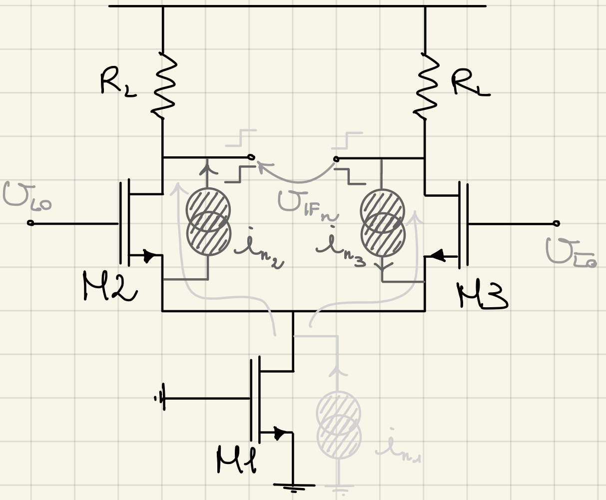
Conversion (voltage) gain: $A_v = \dfrac{V_{IF}(\omega_{LO}-\omega_{RF})}{V_{RF}(\omega_{RF})} = \dfrac{2}{\pi} g_{m1}R_L > 1$

Noise:





unbalanced



balanced

Noise PSD changes whether the circuit is unbalanced (i.e one MOS is fully on and the other is fully off) or balanced (i.e. both transistor are slightly on or off). If the switching time is not instantaneous, then there will be a fraction of the period during which the circuit is balanced.

$$S_{v_{IF}}^{SSB}\Big|_{UNBAL} = \underbrace{2\cdot 4KTR_L}_{\text{2 } R_L \text{ resistors}} + 4KT\,\gamma/\alpha\, g_{m_1} R_L^2 + \underbrace{0}_{\text{M2 and M3 are cascoded}}$$

$v_{IF_n} = R_L \cdot i_{n_1} \cdot x_{LO} \longleftrightarrow S_{v_{IF}}\Big|_{M1} = R_L^2\, S_{i_{M1}} P_{x_{LO}}$ where $P_{x_{LO}} = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_{LO}^2\,dt = 1$

$$S_{v_{IF}}^{SSB}\Big|_{BAL} = \underbrace{8KTR_L}_{R_L} + \underbrace{8KT\,\gamma/\alpha\, g_{m_{23}} R_L^2}_{M2 + M3} + \underbrace{0}_{M1}$$

If abrupt switching: $S_{v_{IF}} \simeq S_{v_{IF}}\Big|_{UNBAL}$

If low-pass filtering at mixer output: $\langle S_{v_{IF}} \rangle \simeq S_{v_{IF}}\Big|_{UNBAL} \cdot \overbrace{\left(1 - \frac{2t_w}{T_{LO}}\right)}^{\text{DC of unbal. config.}}$

low-pass filtering → average → $\langle S_{v_{IF}} \rangle$

$$+ S_{v_{IF}}\Big|_{BAL} \cdot \underbrace{\frac{2t_w}{T_{LO}}}_{\text{DC of bal. config.}}$$

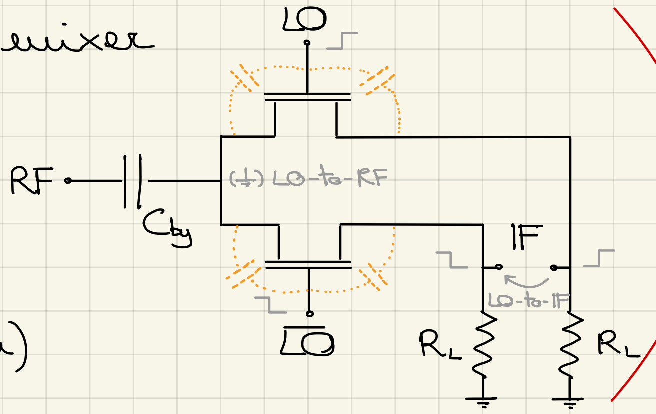
Passive single-balanced mixer

- $A_v = \frac{2}{\pi}\,\frac{R_L}{R_L + r_{on}} = 2A_v\Big|_{RZ} < 1$
- zero RF-to-IF feedthrough

  zero LO-to-RF ″

  non-zero LO-to-IF ″

(same for the active version)



Is there a mixer topology that also has zero LO-to-IF feedthrough?

↓

Double-Balanced mixers

↳ both LO and RF are balanced

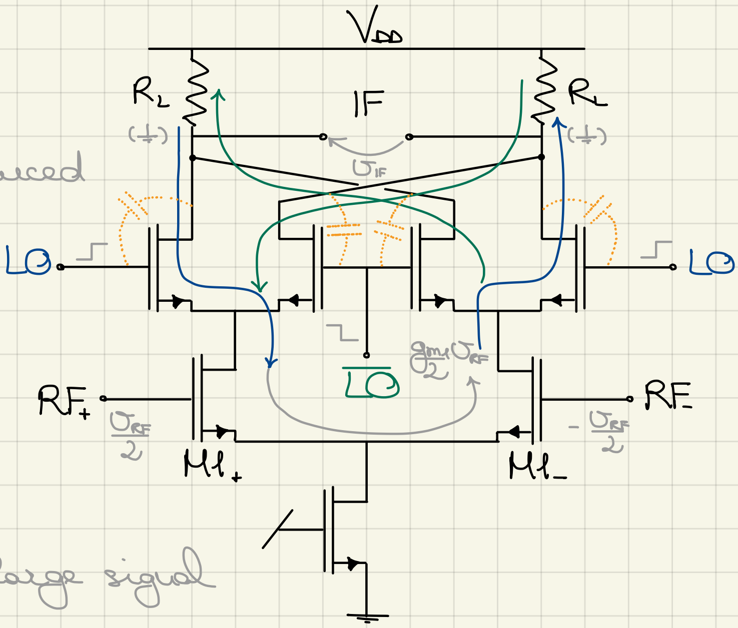
$$v_{IF}(t) = g_{m_1} v_{RF}(t) \cdot x_{LO}(t)$$

$$\Downarrow$$

- $A_v = \frac{2}{\pi}\, g_{m_1} R_L$

- zero LO-to-IF feedthrough

  ↳ valuable since LO is large signal



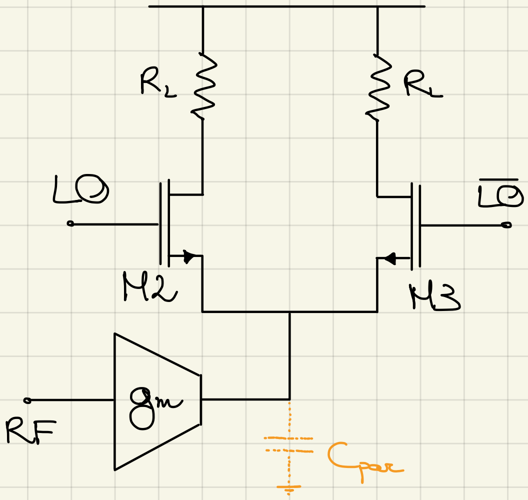
- Linearity (of active mixers):
  - linearity of g_m stage
  - current division between M2/M3 and C_{par}

    ↓

    non-linear if M2/M3 go to triode region

    ↓

    limited LO amplitude



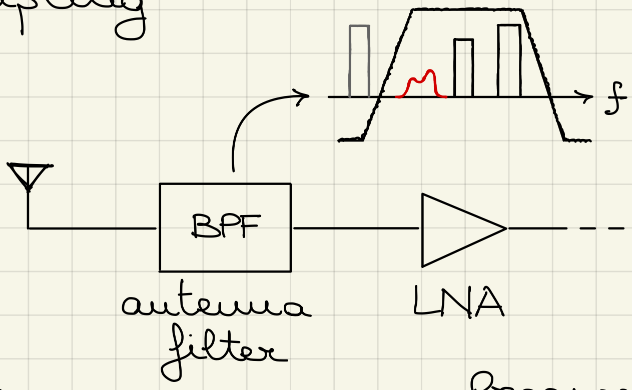
Transceivers Architectures

RX Architectures

- Heterodyne architecture
  - Single IF
  - Double IF

  Direct conversion or Zero-IF architecture
- Sliding IF

  IF sampling



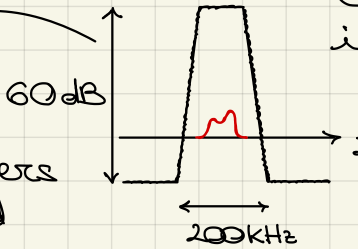
antenna filter attenuates out-of-band interferers

↓

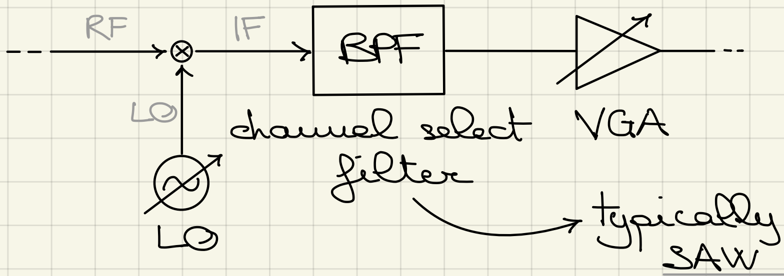
can't perform channel selection

because channel selectivity in wireless systems is in the order of 60dB

the selectivity and bandwidth of such filter can't be feasibly obtained with standard filters (which would also need to be tunable)



A possible solution is using an heterodyne receiver.



→ typically an offchip SAW filter

↓

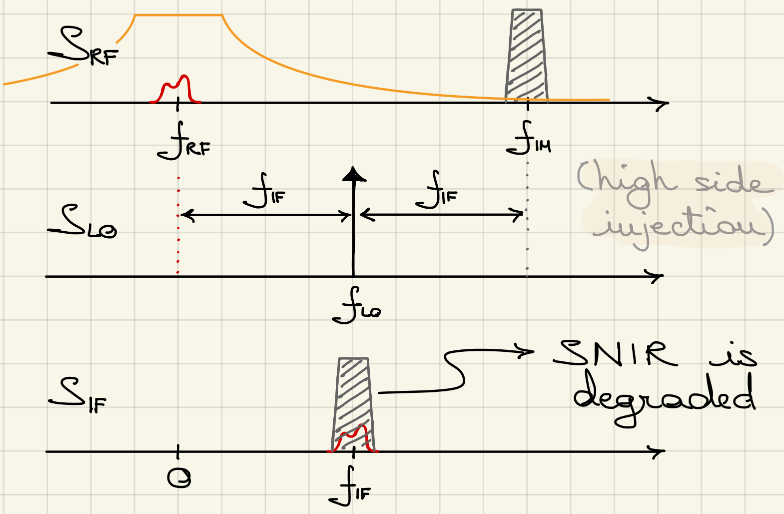
50÷60dB selectivity at center frequency ≃ 10÷100MHz

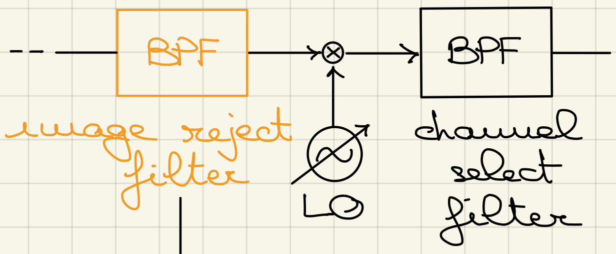
2 advantages:

- IF freq. is lower than RF freq.
- IF filter does not need to be tunable

→ low IF to improve selectivity (lower center freq., lower Q required)

Issue: image problem → Solution





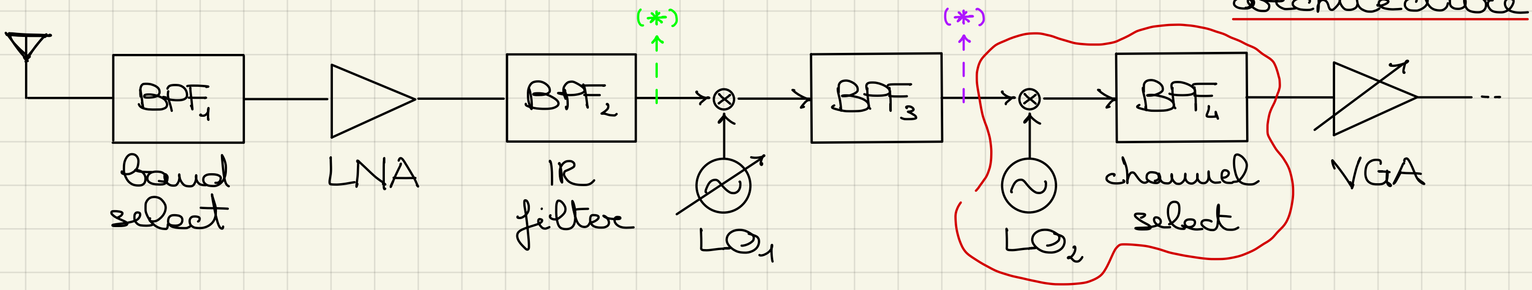
→ high IF to improve image rejection

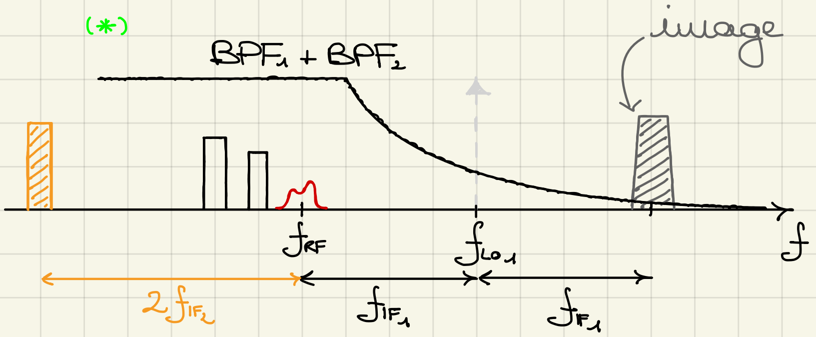
⟹ Trade-off between

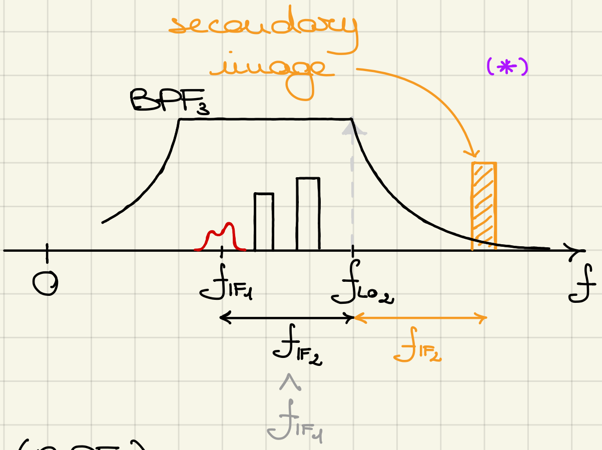
RX sensitivity ↔ images (high IF)

RX selectivity ↔ in-band interferers (low IF)

Another solution to relax this trade-off: Dual-IF architecture







- Large f_{IF_1}: relaxes IR filter (BPF$_2$)
- Small f_{IF_2}: relaxes channel select filter (BPF$_4$)

Issue: secondary image

A signal in the same band of our channel can become an image from the point of view of the second mixer

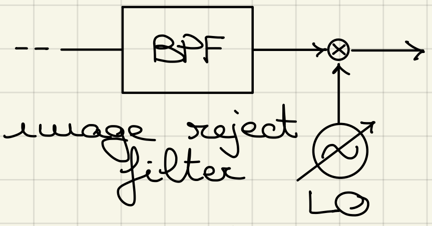
↓

Solution: use BPF$_3$ to filter out the secondary image.

However, now doesn't BPF$_3$ need a larger f_{IF_2} to effectively reject the secondary image?

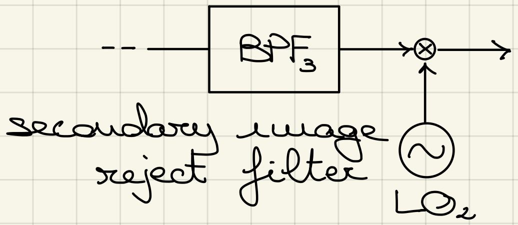
Not necessarily, since the center freq. has been brought down to f_{IF_1} hence the Q factor will be anyway lower (with respect to the IR filter of the single-IF architecture, which was centered around f_{RF})

Single-IF



$$Q \propto \frac{f_{RF}}{2 f_{IF}}$$

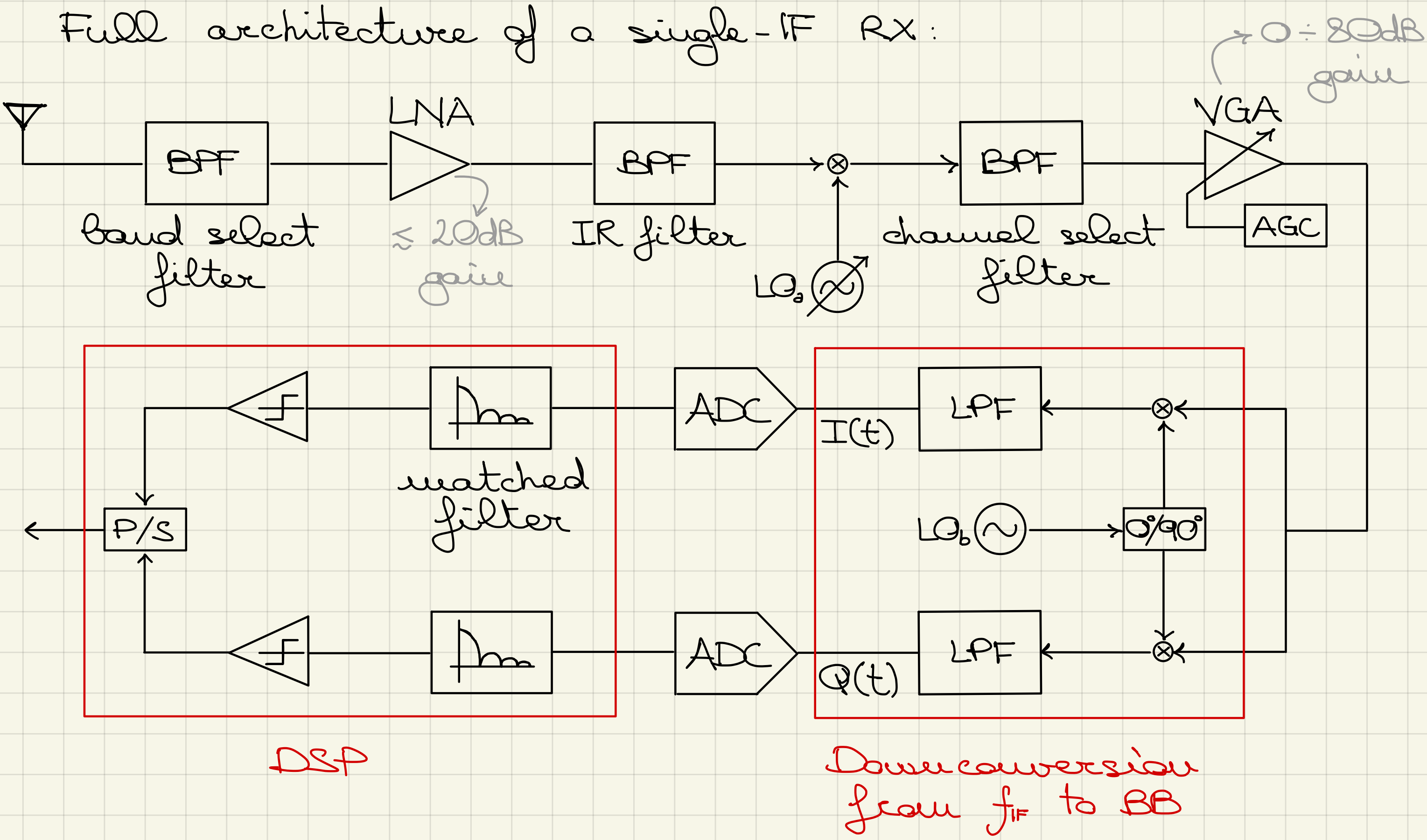
Dual-IF



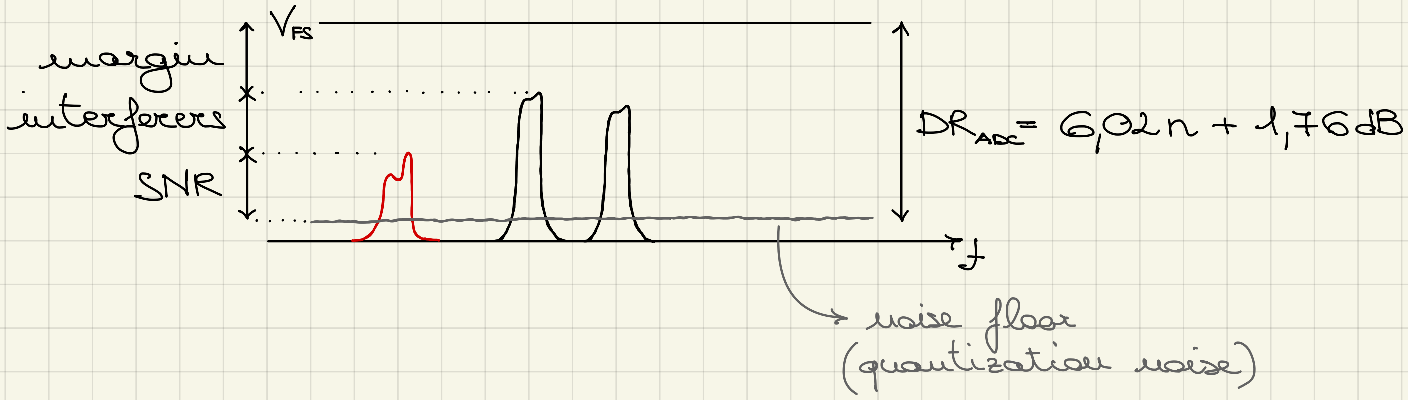
$$Q \propto \frac{f_{IF_1}}{2 f_{IF_2}}$$

These advantages and the elimination of the sensitivity-selectivity trade-off come at the cost of additional components, with additional noise and non-linearities.

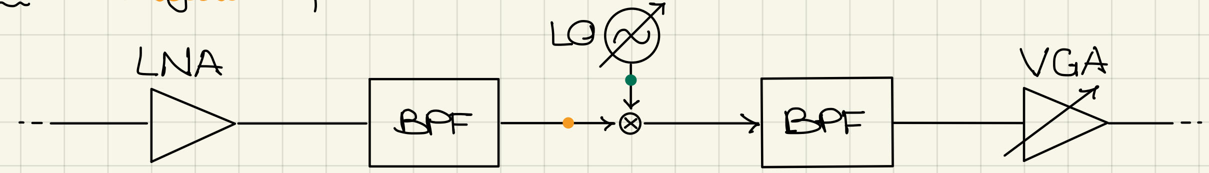
Full architecture of a single-IF RX:

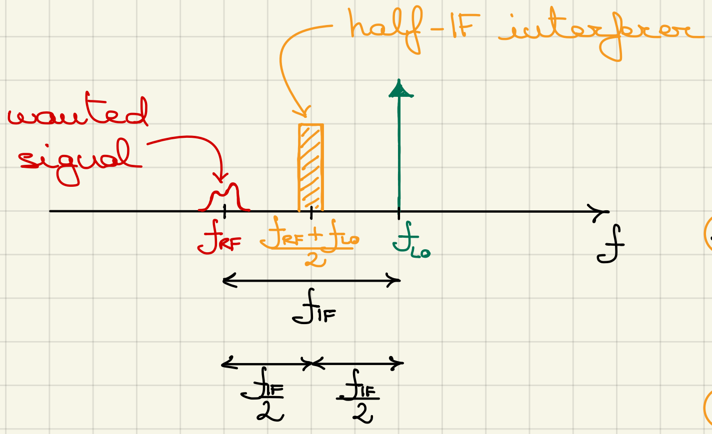


The high variable gain of the VGA is needed to allow any signal, which can span from $-100dB_m$ to $0dB_m$ ($6\mu V$ to $600mV$ peak-to-peak, on a 50Ω resistance) to exploit the FSR of the ADC.
To choose the gain of the VGA, an AGC (Automatic Gain Control) system must check the amplitude of the incoming signal.
Finally, the number of bits and FSR of the ADC must be chosen to account for not only the SNR, but also the presence of interferers which might cause saturation issues, while keeping some margin for possible errors.



Issue: half-IF problem





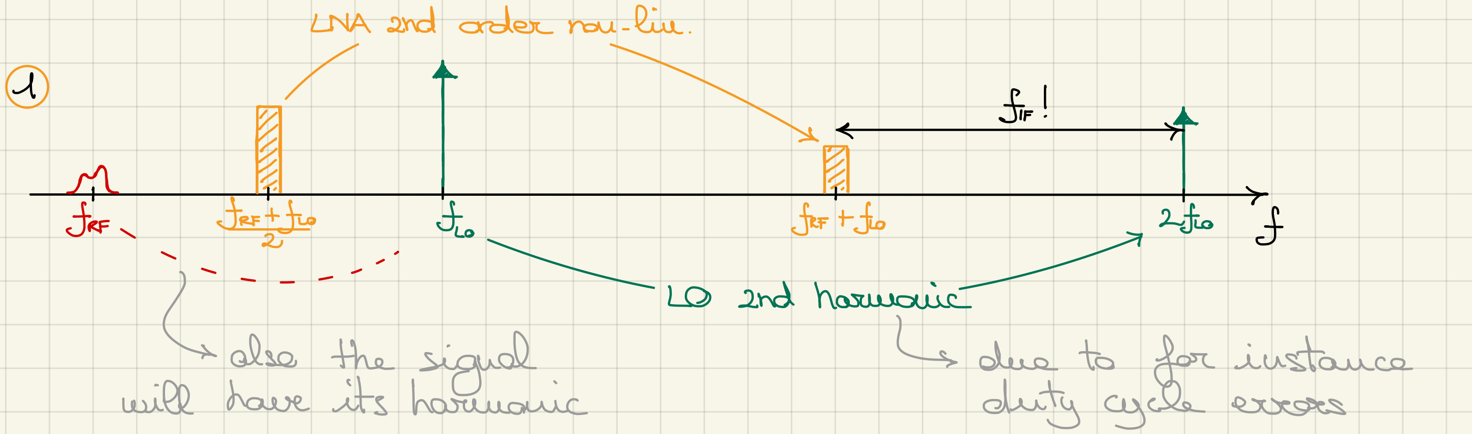
How can this interferer harm?

2 mechanisms:

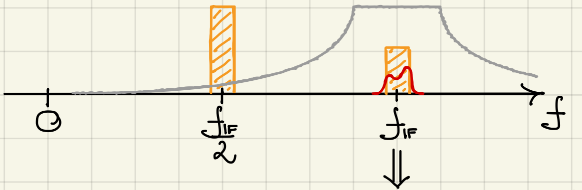
① LO 2nd harmonic
+
LNA 2nd order non-linearity

② VGA 2nd order non-linearity

①



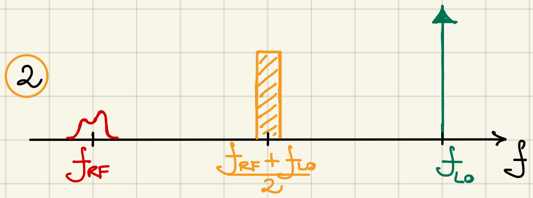
At the output of the mixer:



⇓

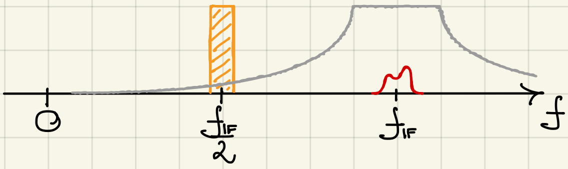
SNIR degraded!

②

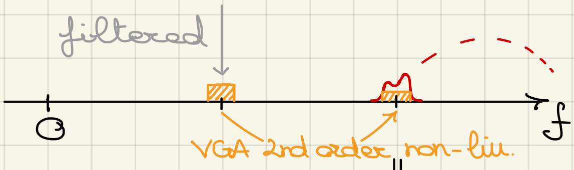


(no harmonics)

At the output of the mixer:



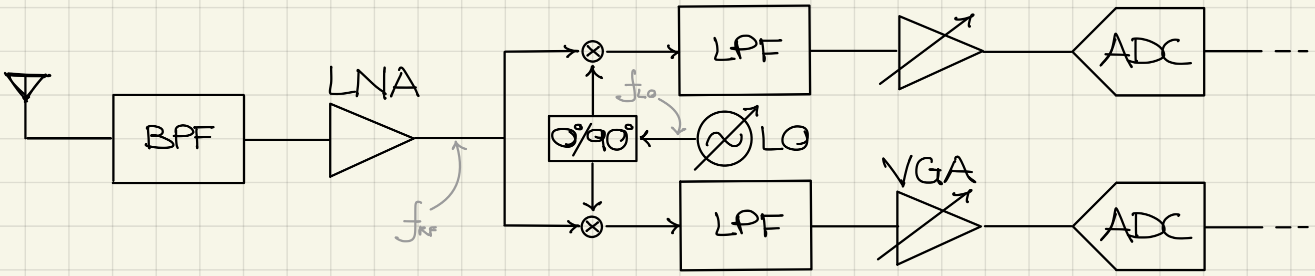
At the output of the VGA:



⇓

SNIR (slightly) degraded!

Direct-Conversion RX (or Zero-IF RX):



$\boxed{f_{RF} = f_{LO}}$ in a zero-IF RX $\Rightarrow$ $\boxed{f_{IF} = |f_{LO} - f_{RF}| = 0}$

The double demodulation is needed to individually recover the transmitted I and Q (this is true for any receiver).

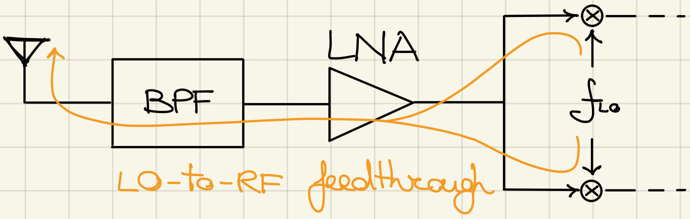
Advantages:

- Image problem apparently solved
  $\Rightarrow$ no need for IR filter
- Channel selection is performed with a LPF (rather than BPF)
  $\Rightarrow$ no need for offchip SAW filters
  LPF can be implemented in silicon (SC active filters)
  SC = switched capacitor

$\Longrightarrow$ Direct-conversion RX architecture suitable for fully integration in silicon

Critical issues:

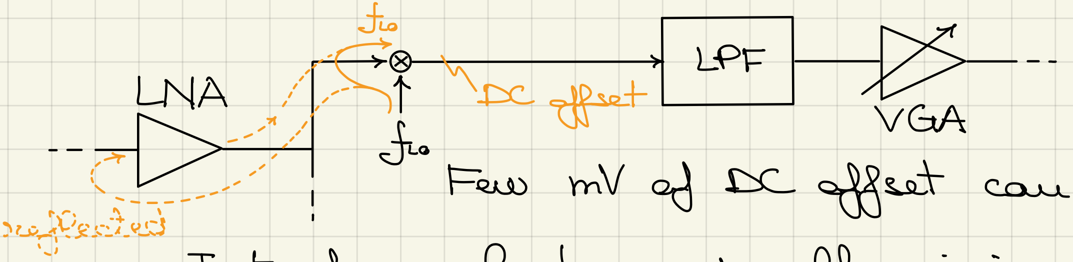
- LO leakage: $f_{LO} = f_{RF} \Rightarrow$ LO is in LNA and BPF BW (BPF = Band select)
  (LO signal also has large power)



$\Downarrow$

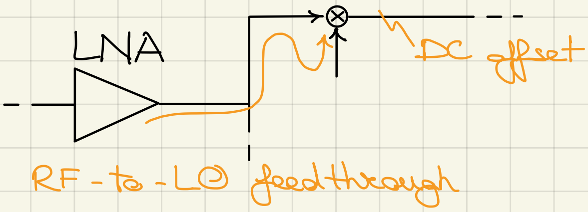
LO signal can be emitted and RX might violate radiation limits (<-80÷-50dBm)

- DC offsets: - LO leakage $\Rightarrow$ self-mixing of LO



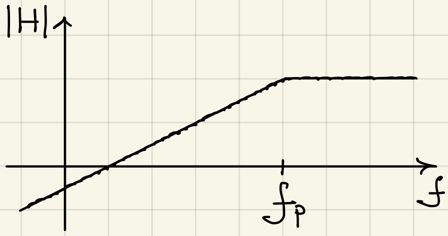
Few mV of DC offset can saturate the VGA

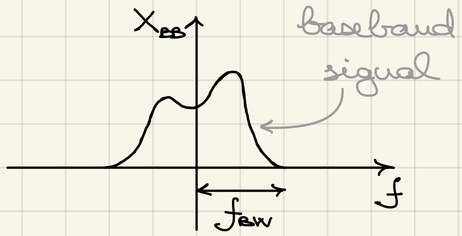
- Interferer leakage $\Rightarrow$ self-mixing of interferer



How can we filter these DC offsets?

1) AC coupling:

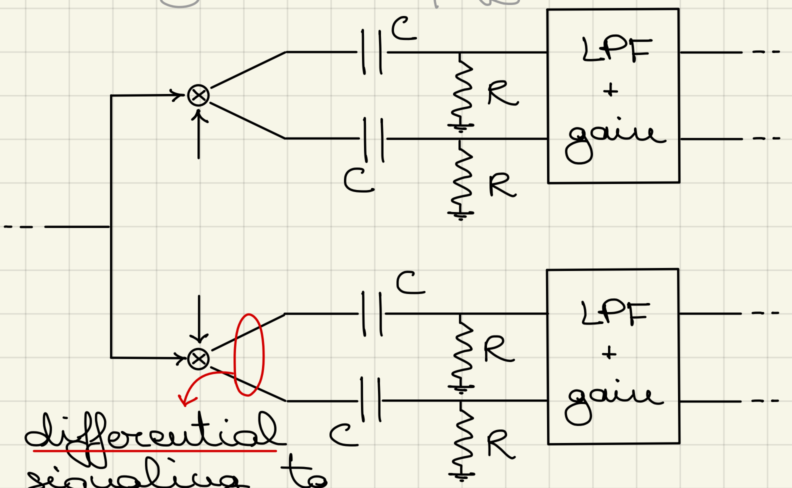




In order to leave most of the signal intact: $f_p < \frac{f_{BW}}{1000}$

(remove only the lowest frequency components)

* noise here is relevant since we have yet to amplify



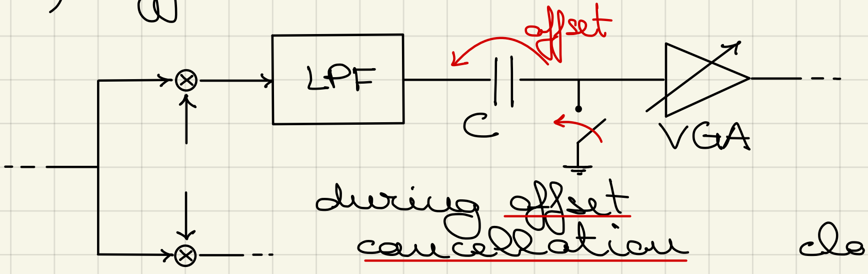
differential signaling to remove 2nd harmonics which would introduce more DC offsets

- AC coupling requires a total of 4 capacitors
- C · R product must be large to have a low f_p
- R resistors introduce noise* (degrading SNR)

⇓

a good implementation would need 4 very large capacitances (to have low R noise)

2) Offset cancellation with switched Capacitor



Since signals come in bursts (e.g TDMA), when no signal is present the switch is closed and the offset is memorized in the capacitor.

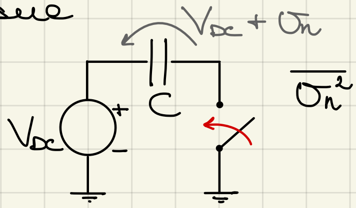
When the signal is present, the switch is opened and the offset cancels out with the voltage drop of the capacitor leaving only the signal at the input of the VGA.

An issue of this solution is that the offset due to interferers might not be constant and cause offset compensation errors; not only that, also DC offsets coming from LO leakage that has been emitted and then reflected back in the RX depend on the surrounding environment and are therefore (slowly) variable in time.

To compensate such errors one can average the offsets sampled over several samples to derive a more correct DC cancellation

The switched capacitor offset cancellation:

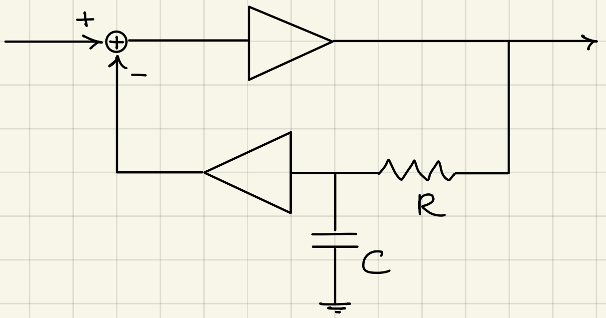
- solves the low-frequency pole issue
- does not solve the noise issue:



$$\overline{v_n^2} = \frac{KT}{C}$$

capacitance still needs to be large to have low noise

3) Offset cancellation with feedback

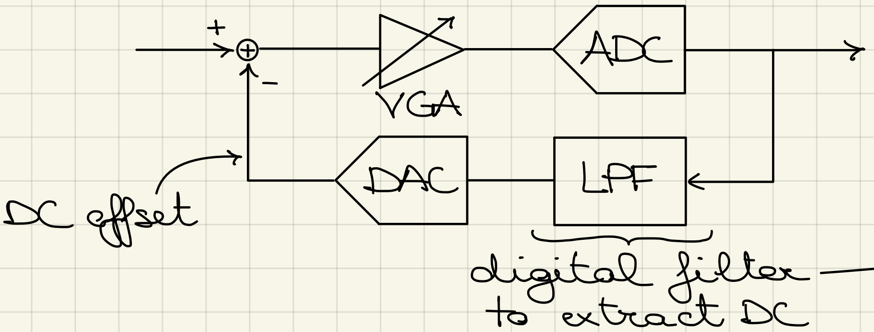


It can be demonstrated that this solution requires a C larger than that of AC coupling

$\Downarrow$

not a viable option

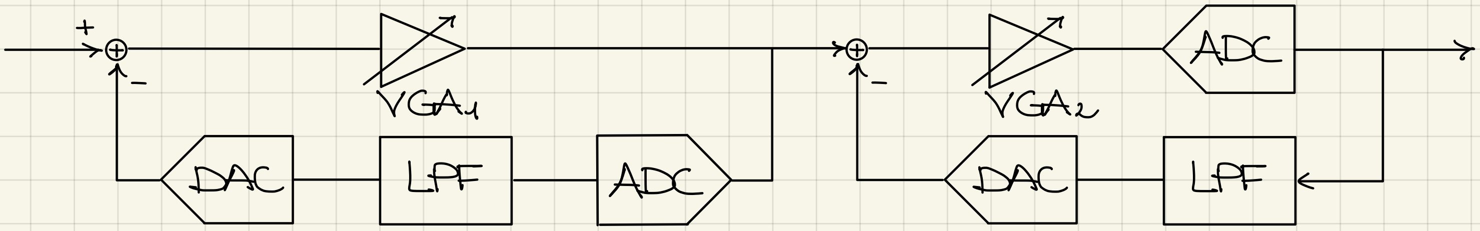
4) Offset cancellation with DAC



digital filter to extract DC

being fully digital, it has no constraints on capacitance sizes

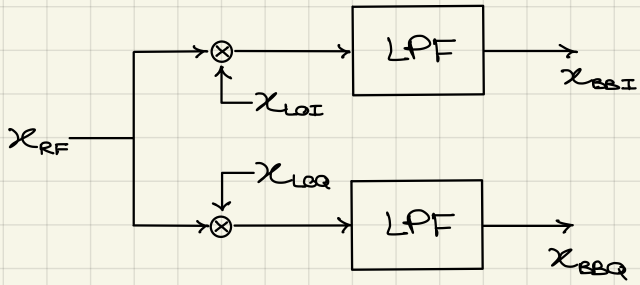
Two-step: to avoid VGA saturation



This is the most-used technique in CMOS technology.

Another critical issue of direct-conversion RX:

- I/Q mismatch



Two paths:

amplitude mismatch ε

phase mismatch ϑ

$$x_{RF}(t) = I(t)\cos\omega_0 t + Q(t)\sin\omega_0 t$$

$$\begin{cases} x_{LOI}(t) = 2\left(1+\frac{\varepsilon}{2}\right)\cos\left(\omega_0 t + \frac{\vartheta}{2}\right) \\ x_{LOQ}(t) = 2\left(1-\frac{\varepsilon}{2}\right)\sin\left(\omega_0 t - \frac{\vartheta}{2}\right) \end{cases}$$

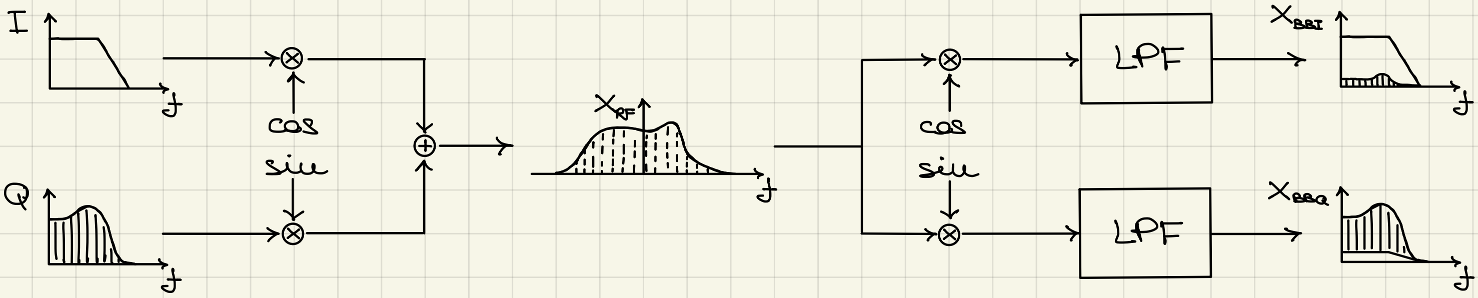
$$\Longrightarrow \begin{cases} x_{BBI}(t) = I(t)\left(1+\frac{\varepsilon}{2}\right)\cos\frac{\vartheta}{2} - Q(t)\left(1+\frac{\varepsilon}{2}\right)\sin\frac{\vartheta}{2} \\ x_{BBQ}(t) = Q(t)\left(1-\frac{\varepsilon}{2}\right)\cos\frac{\vartheta}{2} - I(t)\left(1-\frac{\varepsilon}{2}\right)\sin\frac{\vartheta}{2} \end{cases}$$

wanted signal component   image leakage

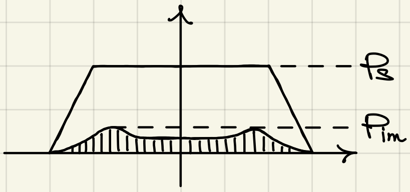
orthogonal component leaking for non-zero ϑ

$\varepsilon \Longrightarrow$ gain error

$\vartheta \Longrightarrow$ crosstalk or image leakage



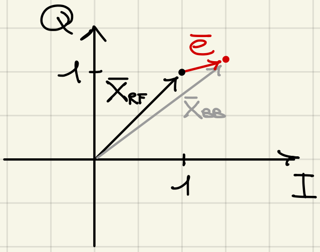
It is still an "image" problem where the image now comes from the same RF frequency of our signal. While in an heterodyne receiver a good IR filter was needed to reject images, in a direct conversion receiver a good quadrature is needed instead.



$$\left[\, IRR = \frac{P_S}{P_{im}} \,\right]$$ Image Rejection Ratio

$$x_{BBI} = I\cdot\left(1+\frac{\varepsilon}{2}\right)\cos\frac{\vartheta}{2} - Q\cdot\left(1+\frac{\varepsilon}{2}\right)\sin\frac{\vartheta}{2}$$

$$x_{BBQ} = Q\cdot\left(1-\frac{\varepsilon}{2}\right)\cos\frac{\vartheta}{2} - I\cdot\left(1-\frac{\varepsilon}{2}\right)\sin\frac{\vartheta}{2}$$



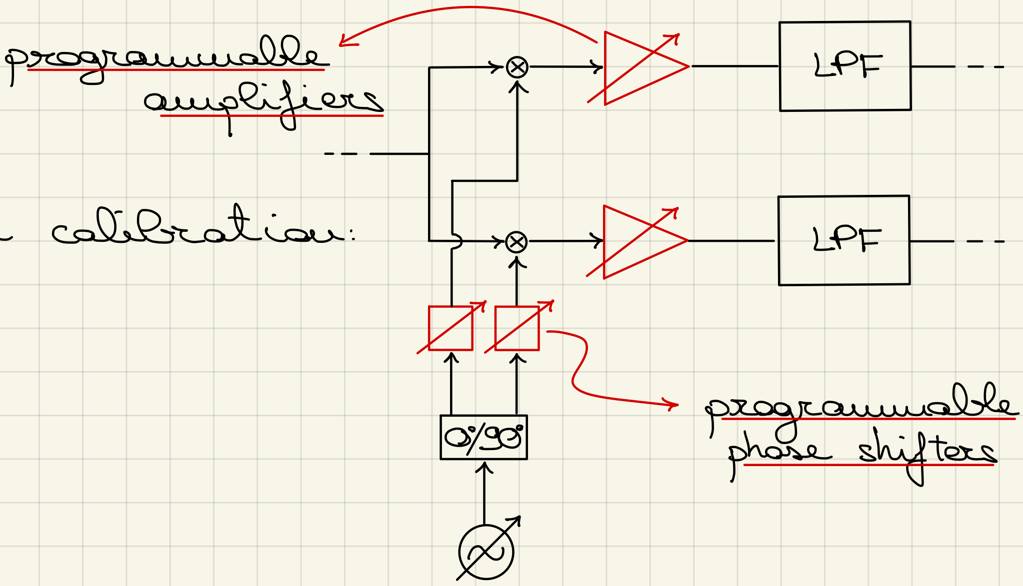
$$IRR = \frac{|\overline{X}_{RF}|^2}{|\overline{e}|^2} = \frac{|\overline{X}_{RF}|^2}{|\overline{X}_{BB} - \overline{X}_{RF}|^2} = \frac{|\overline{X}_{RF}|^2}{(x_{BBI} - I)^2 + (x_{BBQ} - Q)^2} =$$

$$\approx \frac{4}{\varepsilon^2 + \vartheta^2} = \frac{1}{\left(\frac{\varepsilon}{2}\right)^2 + \left(\frac{\vartheta}{2}\right)^2}$$

after due approx.

Typically, an accurate design in GHz range leads to IRR $\simeq$ 30dB (e.g. with $\varepsilon \leq 0{,}1$ and $\vartheta \leq 1°$)

I/Q mismatch calibration:



How come we did not discuss I/Q mismatches in single and dual-IF architectures, since they also have quadrature demodulation? (Note that the other critical issues instead are not present in heterodyne structures).

The reason is that amplitude and, more importantly, phase errors are much weaker when demodulating at low frequencies (i.e. at IF instead of RF)

$x_{LOI}(t)$, $x_{LOQ}(t)$ delay errors $\vartheta = \omega_0 \tau = 2\pi f_0 \tau$

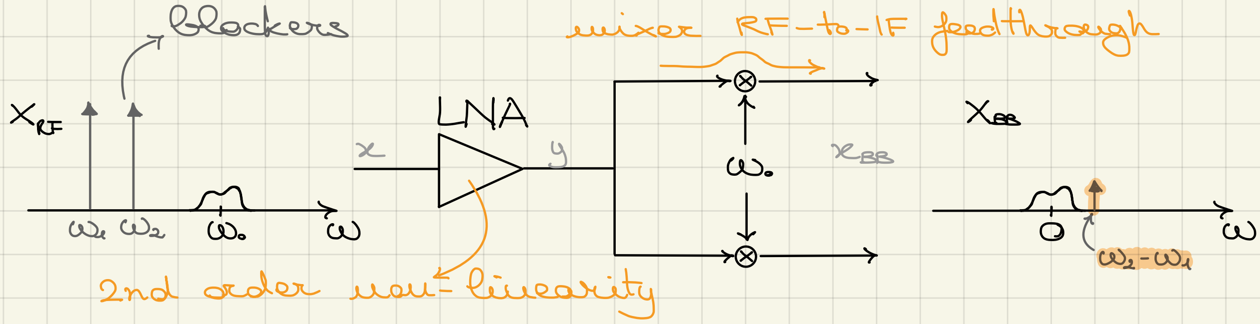
due to time constant and time response mismatch between the two paths

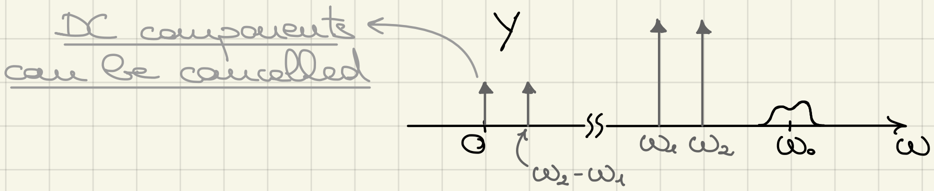
ϑ is actually a function of f_0

$\Longrightarrow$ The larger ω_0, the higher will be the phase error ϑ

Again another critical issue of direct-conversion RX:

- Even-order harmonics



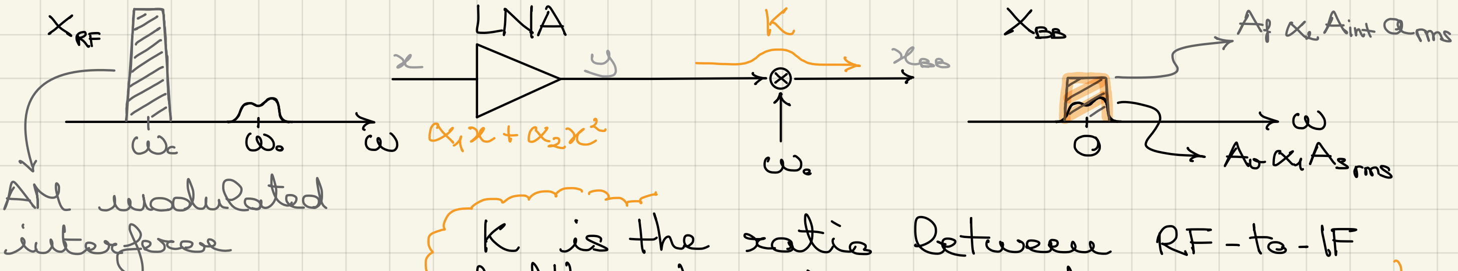


If an interferer is amplitude modulated, another problem arises in the form of unwanted demodulation of AM interferers.

$$x(t) = \underbrace{[A_{int} + a(t)]}_{\text{AM modulation of an interferer}} \cdot \cos\omega_c t \qquad y(t) = \alpha_1 x(t) + \alpha_2 x^2(t)$$

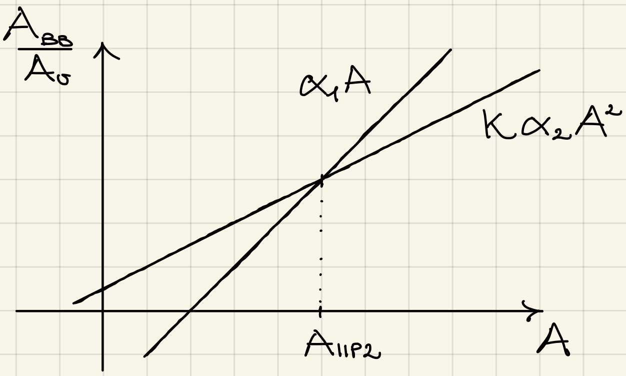
$$\alpha_2 x^2(t) = \alpha_2 \cdot 2 A_{int} \cdot a(t) \cdot \overbrace{\cos^2 \omega_c t}^{\frac{1-\cos(2\omega_c t)}{2}} + \dots =$$

$$= \alpha_2 \cancel{2} A_{int} \cdot a(t) \cdot \frac{1}{\cancel{2}} + \dots$$



K is the ratio between RF-to-IF feedthrough gain A_f and conversion voltage gain of the mixer A_v

e.g.: in a passive RZ mixer $A_f = \frac{1}{2}$, $A_v = \frac{1}{\pi}$
$\longrightarrow K = \frac{1/2}{1/\pi} = \frac{\pi}{2}$



$$SNIR = \frac{A_v \alpha_1 A_{s_{rms}}}{A_f \alpha_2 A_{int} \cdot a_{rms}} = \frac{\alpha_1 A_{s_{rms}}}{K \alpha_2 A_{int} \cdot a_{rms}}$$

$$\alpha_1 A_{IIP2} = K \alpha_2 A_{IIP2}^2 \longrightarrow \frac{\alpha_1}{K\alpha_2} = A_{IIP2}$$

$$= \frac{A_{IIP2} \cdot A_{s_{rms}}}{A_{int} \cdot a_{rms}}$$

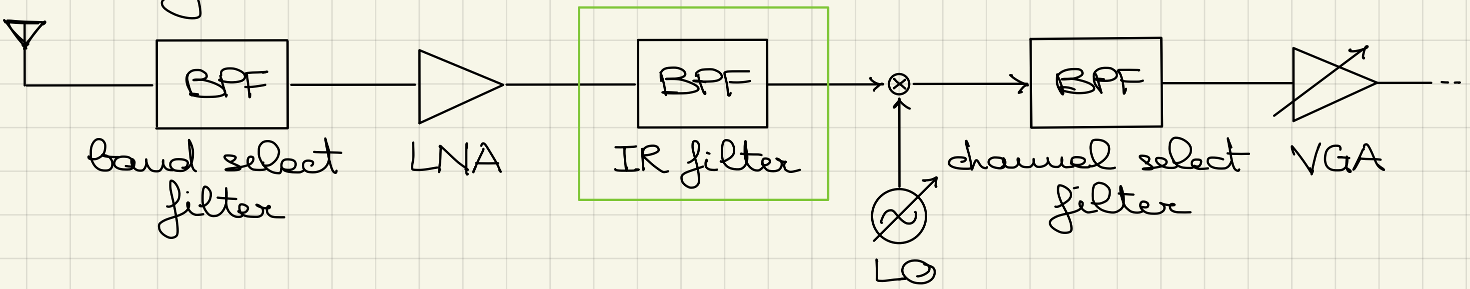
Concluding the list of issues associated with direct conversion architectures, also $1/f$ noise can be especially troublesome due to the fact that the gain stage is at the end of the RX chain and hence all early stages introduce noise that is relevant.

Solutions to this issue are: 1) larger devices to reduce their flicker noise generation and 2) offset cancellation techniques to mitigate the effects of flicker noise at low frequencies.

The direct conversion architecture was the first to be conceived among all RX architectures. However, its several issues made it too hard to be practically implemented and so other solutions (single-IF, double IF) were used.
Only in more recent times was it possible to overcome these issues to exploit the advantages of direct conversion, first of all the possibility of having a fully integrated system.

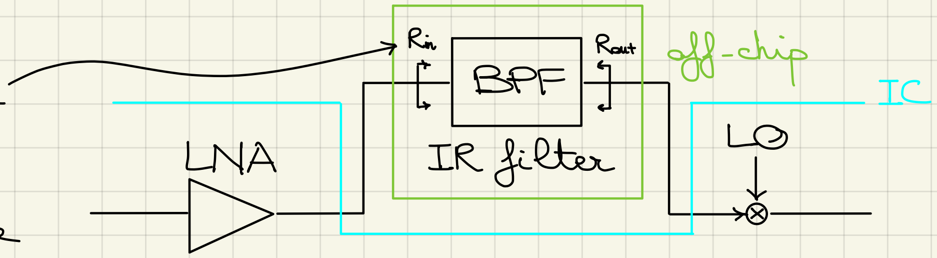
Image-Reject receivers

Single-IF:



Dual-IF: to relax image-selectivity trade-off

These two are solutions based on filtering.
Direct conversion is a solution based on demodulation.
The advantage of the latter solution is that there is no need for additional off-chip filters:

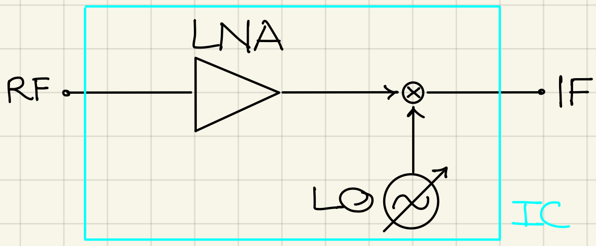


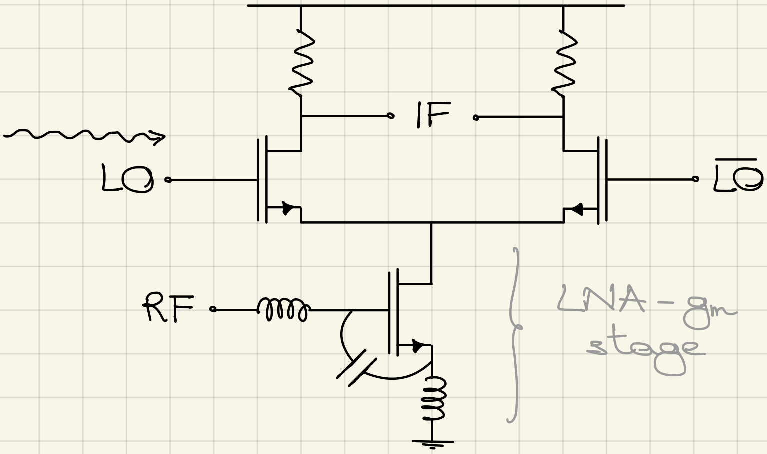
LNA requires an output stage to drive the filter input impedance

RF off-chip blocks (such as filters) require impedance matching

↳ large power consumption

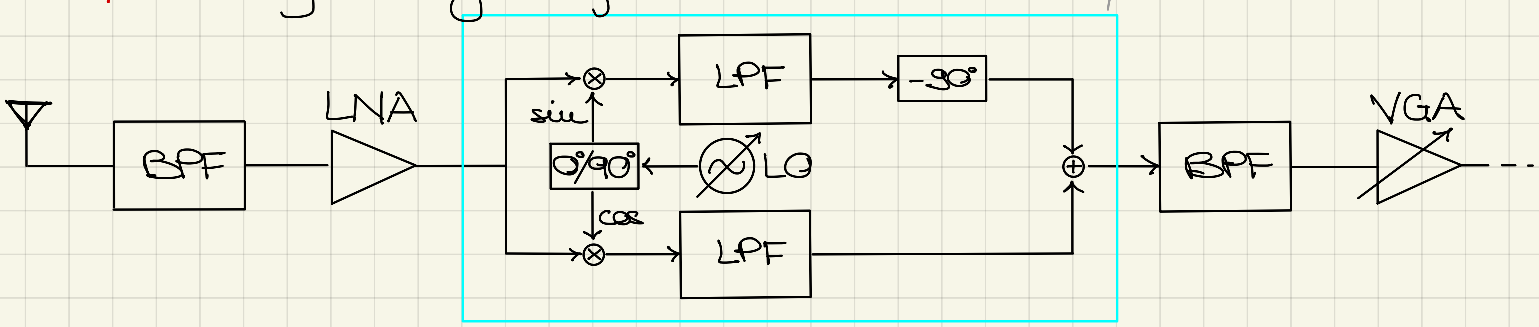
With direct demodulation, instead, the LNA is connected directly to the mixer and requires no impedance matching





Is it possible to pursue image rejection based on filtering, whithout having to deal with BPF in the RF range which would require off-chip blocks?

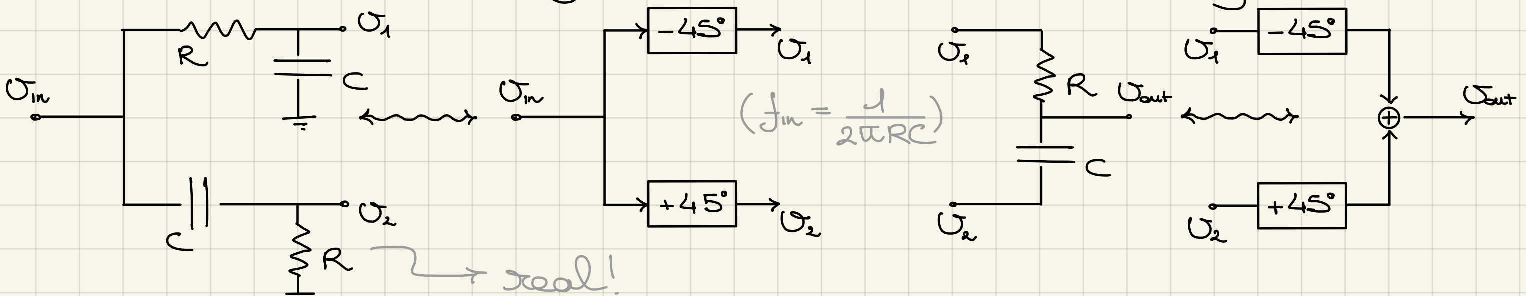
1) Hartley image-reject RX



replaces BPF + mixer

Avoids extra BPF for image rejection. Requires two mixers with quadrature LO, 2 LPFs (which can easily be obtained in integrated circuits unlike BPF at RF) and one phase shifter.

The phase shifter with the summing node can be obtained similarly to what we have already seen:

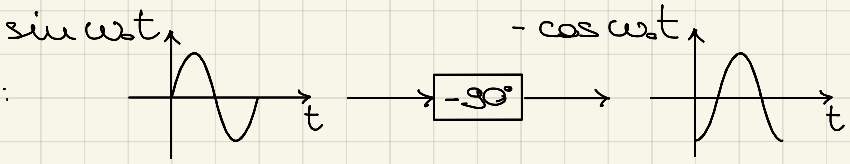


$\left(f_m = \frac{1}{2\pi RC}\right)$

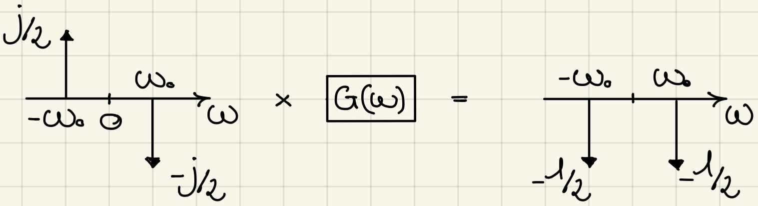
→ real! (introduces losses)

Transfer function of phase shifter → ideal!

In time domain:

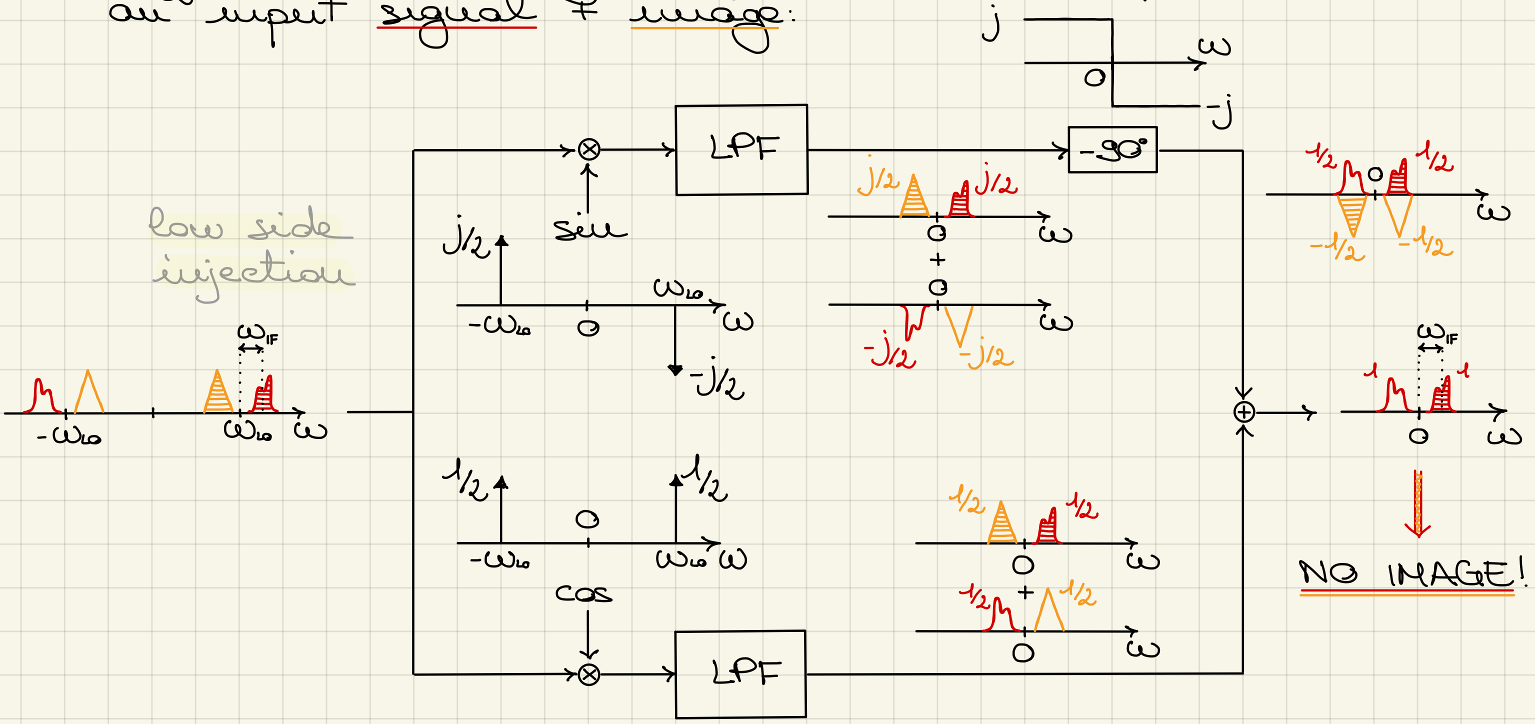


In Fourier domain:



$$\Longrightarrow \boxed{G(\omega) = -j\,\mathrm{sign}(\omega)}$$

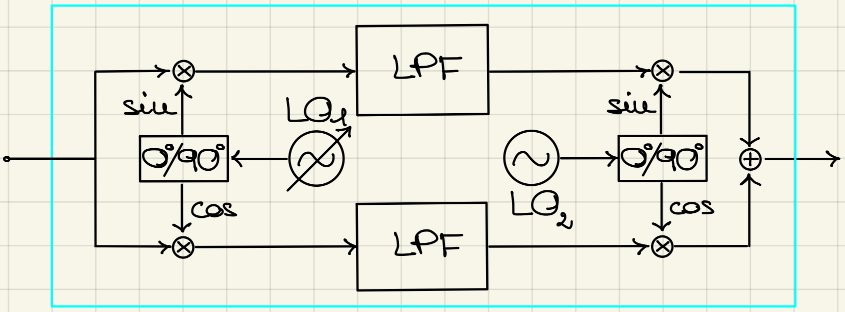
Effects of Hartley IR filter on the spectrum of an input signal + image:



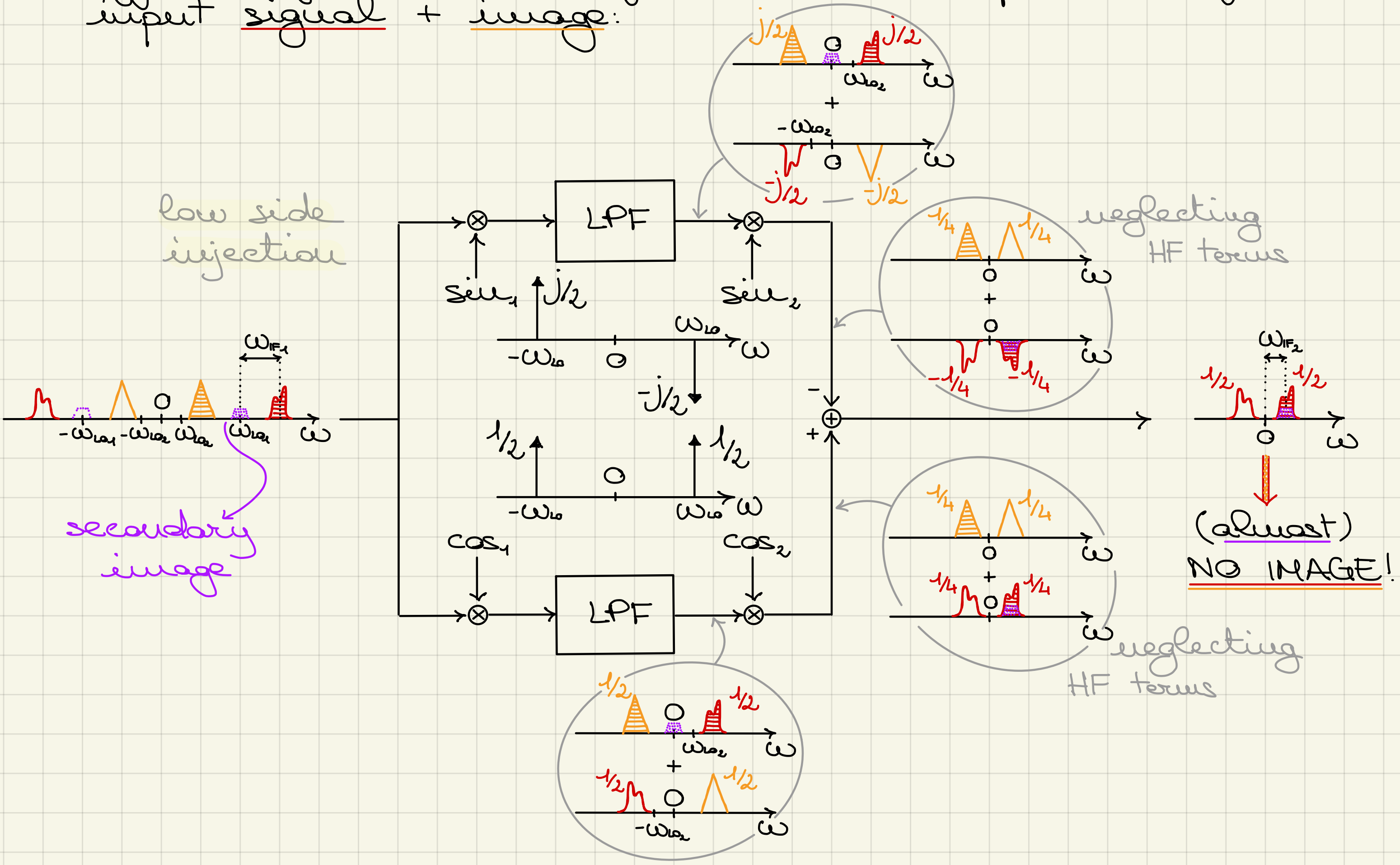
In case of conversion errors (ε and θ) there will be a small leakage of the image in the output spectrum.
The Image Rejection Ratio will be again given by

$$IRR = \frac{4}{\varepsilon^2 + \theta^2}$$

2) Weaver image-reject RX



Effects of Weaver IR filter on the spectrum of an input signal + image:



Comparison of Hartley vs. Weaver architectures

Hartley: phase shifter has limited BW and is sensitive to RC absolute accuracy ⇒ limited IRR
phase shifter also introduces thermal noise and power loss

Weaver: problem of secondary image
⇒ need to use BPF instead of LPF or move ω_{IF_2} to 0

TX Architectures

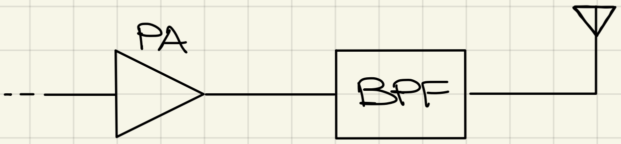
Key issues:

- ACPR: TX has to limit emissions
  ↓
  linearity to avoid spectral regrowth in non-constant envelope modulation

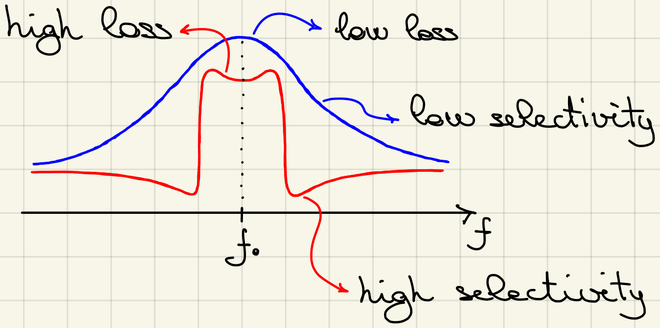
limits PA power efficiency

to have high bit-rate in a limited BW

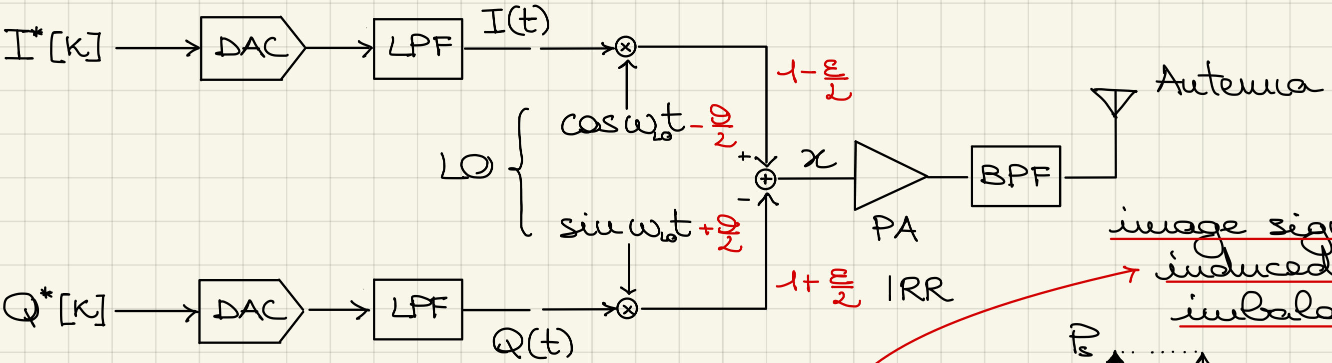
- Loss-selectivity trade-off



power loss degrades efficiency of TX



- Modulation imbalances (ε, ϑ)



Assume $\begin{cases} I(t) = \cos\omega_{BB}t \\ Q(t) = \sin\omega_{BB}t \end{cases}$

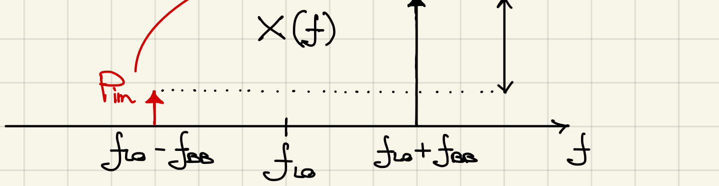


image signal induced by imbalances

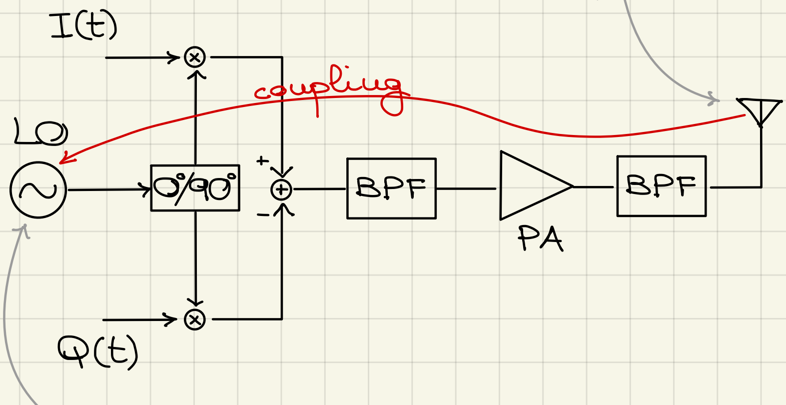
$\Longrightarrow \cos\omega_{BB}t \cdot \cos\omega_{LO}t - \sin\omega_{BB}t \sin\omega_{LO}t = \cos(\omega_{LO}+\omega_{BB})t$ ↳ ideal

With mismatches: $IRR = \frac{P_s}{P_{im}} = \frac{4}{\varepsilon^2+\vartheta^2}$ (no imbalances: $P_{im}=0$)

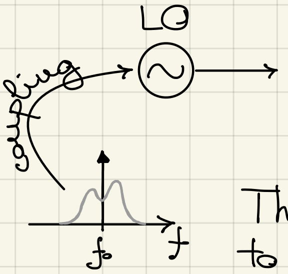
$\Longrightarrow$ Add a BPF before PA to improve IRR

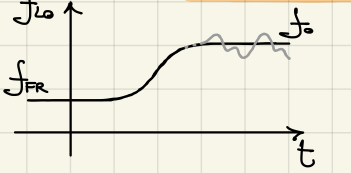
- LO pulling: oscillators are subject to INJECTION LOCKING

signal modulated at f_{LO} with large power



oscillator working at f_{LO}





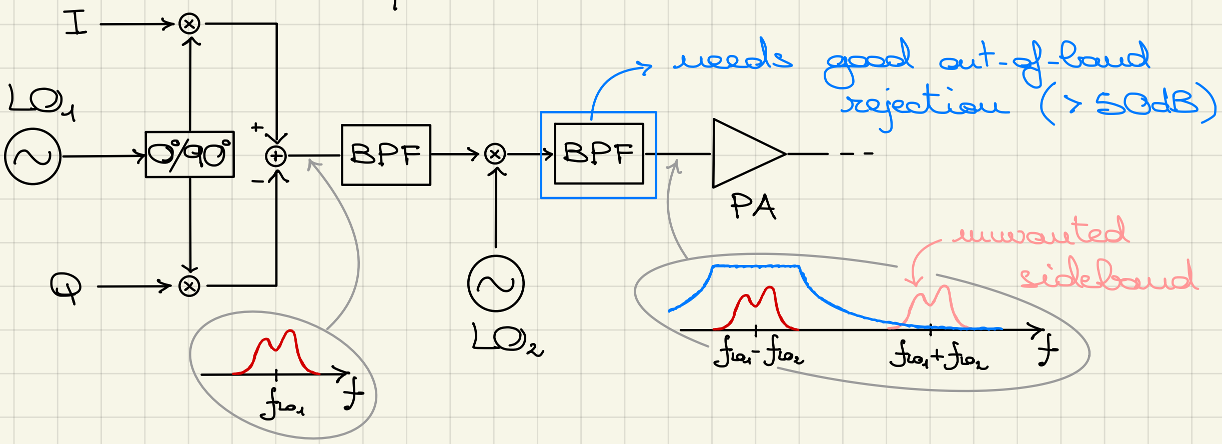
The oscillator "locks" to the injected signal if its frequency is within the oscillator's BW (i.e. 3dB BW of an LC osc.)

If the injected sinusoid is modulated, the oscillator (locked) follows its phase/frequency modulation

→ Introduce an offset between LO frequency and PA frequency to remove coupling

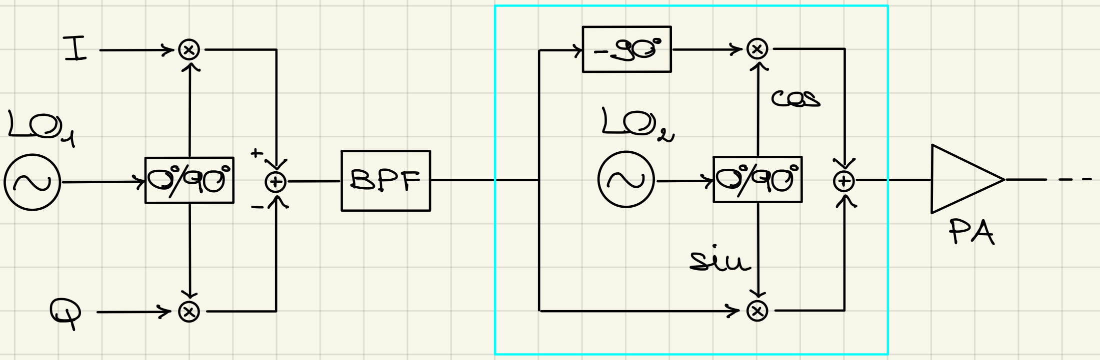
↓

1) Two-step TX architecture



- Avoids LO_1 pulling
- Improves I/Q matching

2) Single-sideband mixer TX architecture



(It is the dual of the Hartley RX architecture)